uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 2: Integration theory

Section 2.2: Integration of general functions

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2 Integration theory

2.2. Integration of general functions

Definition 2.1 Let $(X, \mathcal{A}, \mu)$ be a measure space. The space $L^1(\mu)$ is the collection of all complex measurable functions $f : X \longrightarrow \mathbb{C}$ such that

$$\int_X |f| \, d\mu < \infty .$$

The elements of $L^1(\mu)$ are called *Lebesgue-integrable functions*.

Definition 2.2 *If $f : X \longrightarrow \bar{\mathbb{R}} = [-\infty, \infty]$ is integrable, we define for $E \in \mathcal{A}$*

$$\int_E f = \int_E f^+ - \int_E f^- .$$

If $f : X \longrightarrow \mathbb{C}$ is a measurable function and $f = u + iv$, we call $u, v : X \longrightarrow \mathbb{R}$ the real and imaginary parts of f. If, in addition, $f \in L^1(\mu)$, then we define for $E \in \mathcal{A}$,

$$\int_E f = \int_E u + i \int_E v .$$

Since, $u^+, u^- \leq |u| \leq |f|$ and $v^+, v^- \leq |v| \leq |f|$, the four integrals $\int_E u^+, \int_E u^-, \int_E v^+, \int_E v^-$ are finite and therefore $\int_E f$ is well defined.

Proposition 2.3 *If $f \in L^1(\mu)$, then:* $\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu .$

Corollary 2.4 *$L^1(\mu)$ is a complex vector space, i.e. if $f, g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g \in L^1(\mu)$ and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g , \qquad \forall \, E \in \mathcal{A} .$$

2.2.1. Lebesgue's dominated convergence theorem

Theorem 2.5 (Dominated convergence theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex measurable functions such that $f_n(x) \to f(x)$ as $n \to \infty$ a.e. on X. If there exists a function $F \in L^1(\mu)$ such that*

$$|f_n(x)| \leq F(x) , \qquad \forall \, n \in \mathbb{N}, \ a.e. \, x \in X,$$

then $f \in L^1(\mu)$,

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0 \qquad \text{and} \qquad \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu .$$

This crucial theorem has a lot of consequences. For example:

Corollary 2.6 (Uniform convergence theorem) *Let $(X, \mathcal{A}, \mu)$ be a finite space measure: $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n(x) \to f(x)$ uniformly on X. Then $f \in L^1(\mu)$ and*

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu .$$

Corollary 2.7 *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of measurable functions such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty .$$

Then:

a) The series $\sum_n f_n$ *converges almost everywhere on* X *to a function* $f : X \longrightarrow \mathbb{R}$*:*

$$\sum_{n=1}^{\infty} f_n(x) = f(x)\,, \qquad \textit{for almost all } x \in X\,.$$

b) $f \in L^1(\mu)$*.*

c) $\displaystyle\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu\,.$

2.2.2. Integration with respect to discrete measures

Let $(X, \mathcal{P}(X), \mu)$ be a measure space with X countable, $X = \{x_n\}_{n=1}^{\infty}$ and μ be the discrete measure defined as:

$$\mu(\{x_n\}) = p_n\,, \qquad \mu(A) = \sum_{x_n \in A} p_n\,, \qquad (p_n \geq 0)\,.$$

Let $f : X \longrightarrow \mathbb{C}$ be a complex function.

a) If $f \geq 0$, then $\displaystyle\int_X f\,d\mu = \sum_{n=1}^{\infty} f(x_n) p_n\,.$

b) $f \in L^1(\mu)$ if and only if $\sum_{n=1}^{\infty} |f(x_n)| p_n < \infty$, and in this case,

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} f(x_n)\,p_n\,.$$

2.2.3. Integration with respect to image measures

Let $(X, \mathcal{A}, \mu)$ be a measure space and $\Phi : X \longrightarrow Y$ be a mapping. Let us consider the image measure space $(Y, \mathcal{B}, \nu)$ by Φ ($\mathcal{B} = \Phi(\mathcal{A})$ and $\nu = \mu \circ \Phi^{-1}$) Let $f : Y \longrightarrow \mathbb{C}$ be a function. Then

a) f is $\mathcal{B}$-measurable if and only if $f \circ \Phi$ is $\mathcal{A}$-measurable .

b) If $f \geq 0$ is $\mathcal{B}$-measurable, then $\displaystyle\int_Y f\,d\nu = \int_X (f \circ \Phi)\,d\mu.$

c) If f is $\mathcal{B}$-measurable, then $f \in L^1(\nu)$ if and only if $f \circ \Phi \in L^1(\mu)$, and in this case

$$\int_Y f\,d\nu = \int_X (f \circ \Phi)\,d\mu\,.$$

2.2.4. Integration with respect to measures defined by densities

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\rho : X \longrightarrow [0, \infty]$ be a positive measurable function. Let us consider the measure defined by the density ρ:

$$\nu(A) = \int_A \rho\,d\mu\,, \qquad A \in \mathcal{A}.$$

Then:

a) If $f \geq 0$ is measurable, then $\displaystyle\int_X f\,d\nu = \int_X f\rho\,d\mu.$

b) If $f : X \longrightarrow \mathbb{C}$ is measurable, then: $f \in L^1(\nu)$ if and only if $\displaystyle\int_X |f|\rho\,d\mu < \infty$, and in this case

$$\int_X f\,d\nu = \int_X f\rho\,d\mu\,.$$

2.2.5. Integration with respect to Borel-Stieltjes measures

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing derivable function with bounded derivative g' on each compact set. Let us consider the Borel-Stieltjes measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m_g)$. Then $m_g = g' dm$, that is to say that the Borel-Stieltjes measure m_g coincides with the measure defined by the density g' and therefore for all $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in L^1(m_g)$, we have

$$\int_{\mathbb{R}} f \, dm_g = \int_{\mathbb{R}} f g' \, dm = \int_{\mathbb{R}} f(t) \, g'(t) \, dt \, .$$

2.2.6. Integration with respect to Lebesgue measure on $\mathbb{R}^n$

Let us consider the Lebesgue measure space $(\mathbb{R}^n, \mathcal{M}, m)$ where $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets and m is the Lebesgue measure. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function. Then

a) If $f \geq 0$ or if $f \in L^1(m)$, then

a.1) $\displaystyle\int_{\mathbb{R}^n} f(a+x) \, dx = \int_{\mathbb{R}^n} f(x) \, dx \, .$

a.2) $\displaystyle\int_{\mathbb{R}^n} f(T(x)) \, dx = \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(x) \, dx$, for all $T \in GL(n)$.

a.3) More generally, $\displaystyle\int_A f(T(x)) \, dx = \frac{1}{|\det T|} \int_{T(A)} f(x) \, dx$, for all $T \in GL(n)$ and $A \in \mathcal{M}$.

b) If $\Phi : \mathbb{R} \longrightarrow [0, \infty]$ is a Borel measurable function then

$$\int_{\mathbb{R}^n} \Phi(\|x\|) \, dx = n\Omega_n \int_0^\infty \Phi(r) \, r^{n-1} \, dr \, , \qquad \text{where } \Omega_n = m(\{x \in \mathbb{R}^n : \ \|x\| \leq 1\}) \, .$$

c) Let $B_n = \{x \in \mathbb{R}^n : \ \|x\| < 1\}$, Then

$$\int_{B_n} \frac{dx}{\|x\|^\alpha} < \infty \quad \Leftrightarrow \quad \alpha < n \qquad \text{and} \qquad \int_{\mathbb{R}^n \setminus B_n} \frac{dx}{\|x\|^\alpha} < \infty \quad \Leftrightarrow \quad \alpha > n \, .$$