uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 2: Integration theory

Section 2.4: Decomposition of measures

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2 Integration theory

2.4. Decomposition of measures

Definition 2.1 Let μ, ν be (positive) measures defined on a measurable space $(X, \mathcal{A})$. We say that ν is *absolutely continuous* with respect to μ, and we denote it as $\nu \ll \mu$, if

$$\mu(A) = 0 \implies \nu(A) = 0\,.$$

We say that μ and ν are *mutually singular*, and we denote it as $\mu \perp \nu$, if we can decompose X as $X = A \cup B$ with $A \cap B = \varnothing$ and A is a null-set for μ and B is a null-set for ν.

Theorem 2.2 (Radon-Nikodym theorem for finite measures) *Let μ, ν be finite (positive) measures defined on $(X, \mathcal{A})$. Then, there exists a positive integrable function $f : X \longrightarrow [0, \infty]$, $f \in L^1(\mu)$, and a measure ν_s mutually singular with μ such that*

$$d\nu = f\,d\mu + d\nu_s$$

i.e. for all $E \in \mathcal{A}$

$$\nu(E) = \int_E f\,d\mu + \nu_s(E)\,.$$

Besides, f and ν_s are unique (almost everywhere).

Theorem 2.3 (Radon-Nikodym theorem for σ-finite measures) *If μ and ν are only σ-finite the same is true, but now not necessarily $f \in L^1(\mu)$.*