uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 2: Integration theory

Section 2.5: L^p-spaces

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2 Integration theory

2.5. L^p-spaces

2.5.1.The case $1 \leq p < \infty$

Definition 2.1 A real function $\varphi : (a,b) \longrightarrow \mathbb{R}$ with $-\infty \leq a < b \leq \infty$ is said to be <u>*convex*</u> if

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\,\varphi(x) + \lambda\,\varphi(y)\,, \qquad \forall\, x, y \in (a,b)\,,\ \forall\, \lambda \in [0,1]\,.$$

Example: $\varphi(x) = e^x$.

Theorem 2.2 (Jensen's inequality) *Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$. If $f : X \longrightarrow (a,b)$ is integrable ($-\infty \leq a < b \leq \infty$), i.e. $f \in L^1(\mu)$, and $\varphi : (a,b) \longrightarrow \mathbb{R}$ is a convex function, then*

$$\varphi\Big(\int_X f\,d\mu\Big) \leq \int_X (\varphi \circ f)\,d\mu\,.$$

Example: Take $X = \{p_1, \dots, p_n\}$, $\mu(p_i) = \alpha_1$ with $\sum_i \alpha_i = 1$, $f(p_i) = x_i$, $\varphi(x) = e^x$. Then Jensen's inequality gives

$$e^{\alpha_1 x_1 + \cdots + \alpha_n x_n} \leq \alpha_1 e^{x_1} + \cdots + \alpha_n e^{x_n}\,.$$

and writing $y_i = e^{x_i}$

$$y_1^{\alpha_1} \cdots y_n^{\alpha_n} \leq \alpha_1 y_1 + \cdots + \alpha_n y_n\,.$$

In particular we obtain for $n = 2$

Corollary 2.3 *If $a \geq 0$, $b \geq 0$, $0 \leq \lambda \leq 1$, then $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ with equality if and only if $a = b$.*

Definition 2.4 Let $(X, \mathcal{A}, \mu)$ be a measure space, $0 < p < \infty$. Given a complex function $f : X \longrightarrow \mathbb{C}$ we define

$$\|f\|_p := \Big(\int_X |f|^p\,d\mu\Big)^{1/p}\,,$$

and

$$L^p(X, \mathcal{A}, \mu) = L^p(\mu) = \{f : X \longrightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_p < \infty\}\,.$$

We consider that two functions define the same element of $L^p(\mu)$ when they are equal almost everywhere with respect to μ.

Example: Let $X = \mathbb{N}$ and μ be the counting measure. In this case we denote $L^p(\mu) = \ell^p$:

$$\ell^p = \Big\{\{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^p < \infty\Big\}\,, \qquad \|x_n\|_p = \Big(\sum_{n=1}^\infty |x_n|^p\Big)^{1/p}\,.$$

Proposition 2.5 *$L^p(\mu)$ is a complex vector space.*

Theorem 2.6 (Hölder's inequality). *Suppose that $1 < p < \infty$ and p, q are <u>conjugated exponents</u>, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. If f, g are complex measurable functions on X then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q\,.$$

In particular, if $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then $fg \in L^1(\mu)$ and, in this case, equality holds if and only if $\alpha|f|^p = \beta|g|^q$ a.e. for some $\alpha, \beta \geq 0$, not both of them zero.

Theorem 2.7 (Minkowski's inequality). *If $1 \leq p < \infty$ and $f, g \in L^p(\mu)$ then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

Corollary 2.8 *$L^p(\mu)$ is a normed space for $1 \leq p < \infty$. The number $\|f\|_p$ is called the <u>L^p-norm of f</u>.*

In the case $p = 2$ the norm $\|\cdot\|_2$ comes from the scalar product:

$$\langle f, g\rangle := \int_X f(x)\,\overline{g(x)}\,d\mu(x)\,, \qquad f, g \in L^2(\mu)\,.$$

Therefore, $L^2(\mu)$ is a Hilbert's space and the richer Hilbert's spaces theory applies. Observe that

$$\|f\|_2^2 = \langle f, f\rangle\,, \qquad \forall\, f \in L^2(\mu)\,.$$

2.5.2. The space $L^\infty(\mu)$

Definition 2.9 If $f : X \longrightarrow \mathbb{C}$ is a complex measurable function, we define

$$\|f\|_\infty := \inf\{\alpha \geq 0 :\ \mu(\{x :\ |f(x)| > \alpha\}) = \mu(|f|^{-1}(\alpha, \infty)) = 0\}$$

with the convention $\inf \varnothing = \infty$. The number $\|f\|_\infty$ is called the <u>*essential supremum of* $|f|$</u>.

The infimum is in fact a minimum, because

$$\{x :\ |f(x)| > \alpha\} = \bigcup_n \{x :\ |f(x)| > \alpha + \frac{1}{n}\}$$

and if $\alpha = \|f\|_\infty$ then the sets $\{x :\ |f(x)| > \alpha + \frac{1}{n}\}$ have zero measure.

Observe also that if $\|f\|_\infty \leq K$ then $|f(x)| \leq K$ a.e. on X, and so also

$$\|f\|_\infty = \min\{K > 0 :\ |f(x)| \leq K \text{ a.e.}\}\,.$$

Definition 2.10 $L^\infty(\mu) = \{f : X \longrightarrow \mathbb{C} \mid f \text{ is measurable and } \|f\|_\infty < \infty\}$, with the convention that two functions in $L^\infty(\mu)$ are equal if and only if $f = g$ a.e.

Remark 2.11 1) $L^\infty(\mu)$ depends only on the zero-measure sets of μ. Therefore, if $\nu \ll \mu$ and $\mu \ll \nu$ then $L^\infty(\mu) = L^\infty(\nu)$.
2) Hölder's inequality is trivial for the conjugated exponents 1 and ∞: $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.
3) Since $|f + g| \leq |f| + |g|$, Minkowski's inequality is also trivial for $p = \infty$: $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Corollary 2.12 *$L^\infty(\mu)$ is a complex normed space.*

2.5.3. Completeness

We say that a sequence of measurable functions $\{f_n\}_{n=1}^\infty$ <u>*converges to f in $L^p(\mu)$*</u> if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0\,.$$

We say $\{f_n\}_{n=1}^\infty$ <u>*is a Cauchy sequence in $L^p(\mu)$*</u> if

$$\forall\, \varepsilon > 0,\ \exists N = N(\varepsilon) \text{ such that } \|f_n - f_m\|_p < \varepsilon, \qquad \forall\, n, m > N\,.$$

Theorem 2.13 *$L^p(\mu)$ is a complete metric space for $1 \leq p \leq \infty$, i.e. any Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $L^p(\mu)$ converges in $L^p(\mu)$.*

An interesting corollary of the proof is

Corollary 2.14 *Let $1 \leq p \leq \infty$. If $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(\mu)$ then there exists a subsequence that converges pointwise a.e. to a function $f \in L^p(\mu)$.*
As any convergent sequence is also a Cauchy sequence, if $\{f_n\}_{n=1}^\infty$ converges to f in $L^p(\mu)$ then there exists a subsequence that converges pointwise a.e. to f.

As a consequence of Reverse Minkowski's inequality: $\big|\|f\|_p - \|g\|_p\big| \leq \|f - g\|_p$, we also have

Corollary 2.15 *If $\{f_n\}_{n=1}^{\infty}$ converges to f in $L^p(\mu)$ then $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.*

2.5.4. Density of simple functions

Proposition 2.16 *Let S be the class of all complex measurable simple functions on X such that $\mu(\{x : s(x) \neq 0\}) < \infty$, i.e. such that they are integrable. Then S is dense in $L^p(\mu)$ for $1 \leq p < \infty$. This means that each $f \in L^p(\mu)$ can be approximated in L^p-norm by simple functions in S.*

In the case $p = \infty$ we must consider all simple functions in order to get density:

Proposition 2.17 *The set of all simple functions is dense in $L^\infty(\mu)$.*

Let $C_c(X)$ be the set of continuous functions with compact support, i.e. such that there exists a compact set K such that $f(x) = 0$ for all $x \notin K$. As simple functions on S can be approximated by continuous functions on $C_c(\mathbb{R}^n)$ (Lusin's theorem) we get that:

Theorem 2.18 *$C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, m)$ for $1 \leq p < \infty$.*

This theorem also holds on a large kind of topological spaces with Radon measures.

2.5.5. Duality

If X is a complex linear space, a linear functional on X is a linear map from X to $\mathbb{C}$.
Let μ be a (positive) measure and suppose $1 \leq q \leq \infty$. Let q be the conjugated exponent of p. By Hölder's inequality, for each $g \in L^q(\mu)$ the operator

$$\Phi_g(f) = \int_X f g \, d\mu \,,$$

is bounded on $L^p(\mu)$ and $\|\Phi_g\| := \sup\{|\Phi_g(f)| : \ f \in L^p \,, \ \|f\|_p \leq 1\} \leq \|g\|_q$. The question that naturally arises is: have all bounded linear functionals on $L^p(\mu)$ this form? For $p = \infty$ the answer is negative because $L^1(\mu)$ does not furnish all bounded linear functions on $L^\infty(\mu)$. But, for σ-finite measures, the answer is affirmative for $1 \leq p < \infty$.

Theorem 2.19 *Let $1 \leq p < \infty$, μ be a σ-finite measure on X and Φ be a bounded linear functional on $L^p(\mu)$. Then, there is a unique $g \in L^q(\mu)$, where q is the conjugated exponent of p, such that*

$$\Phi(f) = \int_X f g \, d\mu \,, \qquad \forall \, f \in L^p(\mu) \,,$$

i.e. $\Phi = \Phi_g$. Moreover, $\|\Phi\| = \|g\|_q$.

Therefore, for $1 \leq p < \infty$, the dual space of $L^p(\mu)$, i.e. the space of all bounded linear functionals on $L^p(\mu)$ can be identified with $L^q(\mu)$.