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Departamento de Matemáticas

Integration and Measure

Chapter 3: Integrals depending on a parameter

Section 3.1: Continuity and differentiability

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3 Integrals depending on a parameter

3.1. Continuity and differentiability

Let $(X, \mathcal{A}, \mu)$ be a measure space, $I \subseteq \mathbb{R}$ be an open interval and $f : X \times I \longrightarrow \mathbb{R}$ be a function such that for all $t \in I$ the function $f(\cdot, t) \in L^1(X, \mu)$. Hence we can define a new function

$$F(t) = \int_X f(x,t)\, d\mu(x)\,.$$

Theorem 3.1 (Continuity). *Let t_0 be an accumulation point of I, i.e. there is a sequence $\{t_n\}_{n=1}^{\infty} \subset I$ such that $t_n \to t_0$ as $n \to \infty$. Let us also suppose that*

(1) $\exists \lim_{t \to t_0} f(x,t)$ *a.e.* $x \in X$.

(2) $\exists\, g \in L^1(X, \mu))$ *such that* $|f(x,t)| \leq g(x)$ *a.e.* $x \in X$, *for all* $t \in I$, $t \neq t_0$.

Then

$$\lim_{t \to t_0} F(t) = \int_X \big(\lim_{t \to t_0} f(x,t) \big)\, d\mu(x)\,.$$

Hence, if $\lim_{t \to t_0} f(x,t) = f(x,t_0)$ *a.e.* $x \in X$, *then* $F(t)$ *is continuous at* $t = t_0$.

Theorem 3.2 (Differentiability). *Let us suppose that*

(1) $\exists \frac{\partial f}{\partial t}(x,t)$ *a.e.* $x \in X$ *and for all* $t \in I$.

(2) $\exists\, F \in L^1(X\, \mu)$ *such that* $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq F(x)$ *a.e.* $x \in X$, *for all* $t \in I$.

Then F is derivable on I and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x)\,.$$