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Departamento de Matemáticas

Integration and Measure

Chapter 3: Integrals depending on a parameter

Section 3.2: Fourier transform

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3 Integrals depending on a parameter

3.2. Fourier transform

Given $f : \mathbb{R} \to \mathbb{C}$, $f \in L^1(\mathbb{R}, m)$, we define its <u>Fourier transform</u> as

$$\hat{f}(\omega) = \mathcal{F}\big[f\big](\omega) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx\,.$$

In this case, as $|\hat{f}(\omega)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x)|\,dx = \frac{\|f\|_1}{2\pi} < \infty$, it is clear that $\hat{f} \in L^\infty(\mathbb{R})$.
If $f \notin L^1(\mathbb{R})$ but there exists <u>*the principal value of the integral*</u>, then $\hat{f}$ is defined as this principal value:

$$\hat{f}(\omega) = \mathcal{F}\big[f\big](\omega) \;=\; \text{p.v.}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx := \lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R} f(x)\, e^{i\omega x}\, dx\,.$$

Definition 3.1 *Given $f, g \in L^1(\mathbb{R})$, we define its <u>convolution</u> as*

$$(f * g)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\,.$$

It is easy to check that $\|f * g\|_1 \le \frac{1}{2\pi}\|f\|_1\|g\|_1$ and so $f * g \in L^1(\mathbb{R})$.

Theorem 3.2 (Properties of Fourier transform). *Let $f, g \in L^1(\mathbb{R})$, $\alpha \in \mathbb{R}$, $a, b \in \mathbb{C}$, $I \subseteq \mathbb{R}$ an open interval. Then*

(1) $\mathcal{F}\big[af + bg\big](\omega) \;=\; a\,\mathcal{F}\big[f\big](\omega) + b\,\mathcal{F}\big[g\big](\omega)$.

(2) $\mathcal{F}\big[e^{i\alpha x}f(x)\big](\omega) \;=\; \mathcal{F}\big[f\big](\omega+\alpha)$.

(3) $\mathcal{F}\big[f(x-\alpha)\big](\omega) \;=\; e^{i\alpha\omega}\mathcal{F}\big[f\big](\omega)$.

(4) $\mathcal{F}\big[f(\alpha x)\big](\omega) \;=\; \frac{1}{|\alpha|}\,\mathcal{F}\big[f\big]\left(\frac{\omega}{\alpha}\right)$.

(5) $\mathcal{F}\big[\;\overline{f}\;\big](\omega) = \overline{\mathcal{F}\big[f\big](-\omega)}$.

(6) $\mathcal{F}\big[f * g\big](\omega) \;=\; \mathcal{F}\big[f\big](\omega)\,\mathcal{F}\big[g\big](\omega)$.

(7) *If $xf(x) \in L^1(\mathbb{R})$ then $\hat{f}$ is derivable and*

$$\mathcal{F}\big[xf(x)\big](\omega) \;=\; -i\frac{d}{d\omega}\big(\mathcal{F}\big[f\big](\omega)\big)\,.$$

(8) *If f is derivable, f' is continuous and $f' \in L^1(\mathbb{R})$, then*

$$\mathcal{F}\big[f'\big](\omega) \;=\; -i\omega\,\mathcal{F}\big[f\big](\omega)\,.$$

(9) *If $f(\cdot,t) \in L^1(\mathbb{R})$ for all $t \in I$, $\exists\frac{\partial f}{\partial t}(x,t)$ a.e. $x \in \mathbb{R}$ and for all $t \in I$, and $\exists\; F \in L^1(\mathbb{R})$ such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \le F(x)$ a.e. $x \in \mathbb{R}$ and for all $t \in I$, then*

$$\mathcal{F}\Big[\frac{\partial f}{\partial t}(x,t)\Big](\omega) \;=\; \frac{\partial}{\partial t}\big(\mathcal{F}\big[f\big](\omega)\big)\,.$$

A direct consequence of (8) is the following

Corollary 3.3 *If f is derivable, f' is continuous and $f, f' \in L^1(\mathbb{R})$, then $\lim\limits_{|\omega|\to\infty}\hat{f}(\omega) = 0$.*

This is a weak version of the following result:

Lemma 3.4 (Riemann-Lebesgue lemma). *If $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$, i.e. $\hat{f}$ is continuous and vanishes at infinity:*

$$\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0\,.$$

The following result shows that the Fourier transform can be inverted and that its inverse is essentially a Fourier transform:

Theorem 3.5 (Inversion of Fourier transform). *If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = \mathcal{F}^{-1}\big[\hat{f}\,\big](x) \;:=\; \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega x}\, d\omega\,.$$

In particular, $f \in C_0(\mathbb{R})$.

3.2.1. Fourier transform on $L^2(\mathbb{R})$

As $C_c(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $C_c(\mathbb{R})$ is dense on $L^2(\mathbb{R})$ we have that also $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense on $L^2(\mathbb{R})$ and so, given $f \in L^2(\mathbb{R})$,

$$\exists\, \{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ such that } \|f_n - f\|_2 \to 0 \text{ as } n \to \infty.$$

Now, the definition of $\hat{f}(\omega)$ applies for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. It turns out that in this case $\hat{f} \in L^2(\mathbb{R})$ and, in fact, Plancherel's theorem holds:

$$\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}}\, \|f\|_2\,, \qquad \forall\, f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\,,$$

i.e. Fourier transform preserves L^2-norm in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ up to a multiplicative constant. This fact allows us to extend Fourier transform to the whole $L^2(\mathbb{R})$: Given $f \in L^2(\mathbb{R})$ let $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a sequence such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. But then $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence and as

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|\mathcal{F}(f_n - f_m)\|_2 = \frac{1}{\sqrt{2\pi}}\, \|f_n - f_m\|_2$$

and so we also have that $\{\hat{f}_n\}_{n=1}^{\infty}$ is also a Cauchy sequence and therefore converges to a function $g \in L^2(\mathbb{R})$ since $L^2(\mathbb{R})$ is complete. We define $\hat{f} = g$. But, as $\|\hat{f}_n - g\|_2 \to 0$ as $n \to \infty$, we also have

$$\|f_n\|_2 \to \|f\|_2\,, \qquad \|\hat{f}_n\|_2 \to \|g\|_2\,, \qquad \|\hat{f}_n\|_2 = \frac{1}{\sqrt{2\pi}}\, \|f_n\|_2\,.$$

Hence, as the limit of a sequence is unique, we obtain Plancherel's theorem for all functions in $L^2(\mathbb{R})$:

$$\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}}\, \|f\|_2\,, \qquad \forall\, f \in L^2(\mathbb{R})\,.$$

Theorem 3.6 *One can associate to each $f \in L^2(\mathbb{R})$ a function $\hat{f} \in L^2(\mathbb{R})$ so that the following properties hold:*

(1) *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f}$ is the previously defined Fourier transform.*

(2) *For every $f \in L^2(\mathbb{R})$, $\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}}\, \|f\|_2$, i.e.*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \big|f(x)\big|^2 dx = \int_{-\infty}^{\infty} \big|\hat{f}(\omega)\big|^2 d\omega\,. \qquad \textbf{(Plancherel's theorem)}$$

(3) *The mapping $f \mapsto \hat{f}$ is a Hilbert's space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$: $\frac{1}{2\pi}\,\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, i.e.*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, d\omega\,, \qquad \forall\, f, g \in L^2(\mathbb{R})\,.$$

(4) *The following symmetric relation exists between f and $\hat{f}$. If:*

$$\varphi_R(\omega) = \frac{1}{2\pi} \int_{-R}^{R} f(x)\, e^{i\omega x}\, dx\,, \qquad \psi_R(\omega) = \int_{-R}^{R} \hat{f}(\omega)\, e^{-i\omega x}\, d\omega\,,$$

then

$$\|\varphi_R - \hat{f}\|_2 \to 0 \quad \textit{and} \quad \|\psi_R - f\|_2 \to 0 \qquad \textit{as } R \to \infty\,.$$