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Departamento de Matemáticas

Integration and Measure

Chapter 3: Integrals depending on a parameter

Section 3.3: Laplace transform

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3 Integrals depending on a parameter

3.3. Laplace transform

Given a complex function $f \in L^1(0,\infty)$, let us consider its Fourier transform with argument $\omega \in \mathbb{C}$. As $|e^{i\omega x}| = e^{-\beta x}$ if $\omega = \alpha + i\beta$, the following integral converges

$$\int_0^\infty |f(t)\, e^{i\omega t}|\, dt = \int_0^\infty |f(t)|\, e^{-\beta t} < \infty \qquad if\ \beta > 0.$$

If we restrict to the half-plane $\mathrm{Im}\,(\omega) > a > 0$ we can weaken the requirement that f be integrable, since:

$$\int_0^\infty |f(t)\, e^{i\omega t}|\, dt \le \int_0^\infty |f(t)|\, e^{-at} < \infty$$

if f has *exponential growth* and f is, for example, *piecewise continuous.* In this situation, it is customary to make the change of variable $z = -i\omega$ and to define the Laplace transform of f to be $\mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-zt} dt = 2\pi\, \hat{f}(iz)$.

Definition 3.1 Let $\mathcal{E}$ be the class of complex functions $f : (0,\infty) \longrightarrow \mathbb{C}$ such that

(1) f is integrable on $[0,T]$ for all $T > 0$ (this condition holds, for example if f is piecewise continuous).

(2) f has *exponential growth*: $\lim_{t\to\infty} f(t)\, e^{-\alpha t} = 0$ for some $\alpha \in \mathbb{R}$.

Definition 3.2 If $f \in \mathcal{E}$, then we define its *Laplace transform* as

$$\mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-zt} dt = 2\pi\, \hat{f}(iz)\,.$$

This integral converges for $\mathrm{Re}\, z > \alpha$.

Lemma 3.3 (Riemann-Lebesgue lemma for Laplace transform). *Let $f \in \mathcal{E}$. Then*

a) $\mathcal{L}f(x+iy) \to 0$ as $|y| \to \infty$ for each fixed $x > \alpha$.

b) $\mathcal{L}f(x+iy) \to 0$ as $x \to \infty$ for each fixed y.

Theorem 3.4 (Properties of Laplace transform). *Let $f, g \in \mathcal{E}$, $\alpha, \beta \in \mathbb{C}$, $a \in \mathbb{R}$.*

(1) $\mathcal{L}[\alpha f + \beta g](z) = \alpha\, \mathcal{L}[f](z) + \beta\, \mathcal{L}[g](z)$.

(2) $\mathcal{L}[e^{at} f(t)](z) = \mathcal{L}[f](z-a)$.

(3) $\mathcal{L}[f(at)](z) = \dfrac{1}{a}\, \mathcal{L}[f(t)](z/a), \quad (a > 0)$.

(4) $\mathcal{L}[f(t-a)H(t-a)](z) = e^{-az} L[f](z)$, *where $a > 0$ and $H(t)$ is the Heaviside function:*

$$H(t) = \begin{cases} 0\,, & if\ x \le 0\,, \\ 1\,, & if\ x > 0\,. \end{cases}$$

(5) *If f is continuous and f' is piecewise continuous on $(0,\infty)$, and $f, f' \in \mathcal{E}$, then*

$$\mathcal{L}f'(z) = z\, \mathcal{L}f(z) - f(0)\,.$$

(6) *If $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $(0,\infty)$, and $f, f', \ldots, f^{(n)} \in \mathcal{E}$, then*

$$\mathcal{L}[f^{(n)}](z) = z^n\, \mathcal{L}f(z) - z^{n-1} f(0) - z^{n-2} f'(0) - \cdots - z f^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

(7) $\mathcal{L}f(z)$ *is infinitely derivable on* $\operatorname{Re} z > \alpha$ *if* $f(t), t^n f(t) \in \mathcal{E}$ *(both with exponential growth* α *and*

$$\frac{d^n}{dz^n}[\mathcal{L}f(z)] = (-1)^n \mathcal{L}[t^n f(t)](z)\,.$$

(8) *If* $f \in \mathcal{E}$ *then* $g(t) = \int_0^t f(x)\,dx \in \mathcal{E}$ *and* $\mathcal{L}g(z) = \frac{1}{z}\,\mathcal{L}f(z)$.

(9) *If* $f(t)/t$ *is integrable on* $[0,T]$ *for all* $T > 0$, *then* $\mathcal{L}[\frac{f(t)}{t}](z) = \int_z^\infty \mathcal{L}f(z)\,dz$.

Remark 3.5 In (9) $\int_z^\infty$ denotes integration over any curve in the z-plane starting at z and such that along the curve $\operatorname{Im} z$ stays bounded and $\operatorname{Re} z \to \infty$.

Definition 3.6 (Convolution for Laplace transform). If $f, g \in \mathcal{E}$ then the <u>*convolution*</u> of f and g is defined as

$$(f * g)(t) = \int_0^t f(x)\,g(t-x)\,dx\,.$$

Proposition 3.7 $f * g \in \mathcal{E}$ *and* $\mathcal{L}[f * g](z) = \mathcal{L}f(z)\,\mathcal{L}g(z)$.

3.3.1. Inverse Laplace transform

We want to solve the equation $\mathcal{L}f(z) = F(z)$ where $F(z)$ is a known function. It is clear that the solution, if exists, is not unique since changing the value of f at a countable set does not change the value of $\mathcal{L}f(z)$. However, we have

Theorem 3.8 (Lerch's theorem). *If* f *and* g *are two different continuous functions on* $(0,\infty)$ *such that their Laplace transforms exist, then* $\mathcal{L}f \neq \mathcal{L}g$.

Definition 3.9 Given a derivable function $F(z)$ on $\operatorname{Re} z > \alpha$, we define its <u>*inverse Laplace transform*</u> as the unique continuous function $f : (0,\infty) \longrightarrow \mathbb{C}$ such that $\mathcal{L}f = F$.

Theorem 3.10 (Mellin's inversion formula) *Let* $F(z)$ *be a derivable function on the half-plane* $\operatorname{Re} z > \alpha$ *such that* $F = \mathcal{L}f$ *with* $f : (0,\infty) \longrightarrow \mathbb{C}$ *continuous. Then*

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{zt} F(z)\,dz = \lim_{R\to\infty}\int_{\Gamma_R} e^{zt} F(z)\,dz$$

where Γ_R *is the vertical segment* $\{x + iy :\ |y| \le R\}$ *oriented from* $x - iR$ *to* $x + iR$.