INTEGRATION AND MEASURE | INTERMEDIATE CONTROL 1 - SOLUTIONS

Problem 1 (4 points) Let $(X, \mathcal{A}, \mu)$ be a measure space.

1) Give the definition of σ-algebra and measure.

Prove that:

2) If $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, then $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$.

3) If $A, B \in \mathcal{A}$, then $A \setminus B \in \mathcal{A}$.

4) If $A, B \in \mathcal{A}$, $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and, if $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

5) If $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, then $\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

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1) A collection $\mathcal{A}$ of subsets of a set X is said to be a *σ-algebra* on X, if

(a) $\varnothing \in \mathcal{A}$.

(b) If $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$.

(c) If $\{A_j\}_{j \in \mathbb{N}}$ is a countable collection of members of $\mathcal{A}$, then $\cup_{j=1}^{\infty} A_j \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a *measurable space* and the members of $\mathcal{A}$ are called *measurable sets.* Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ is called a (*positive*) *measure* on X if the following two conditions hold:

a) $\mu(\varnothing) = 0$.

b) μ is countably additive, i.e. if $\{A_i\}_{i=1}^{\infty}$ is a disjoint countable collection of members of $\mathcal{A}$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i)\,.$$

We also say that $(X, \mathcal{A}, \mu)$ is a *measure space*.

2) As $A_j \in \mathcal{A}$ then by properties (b) and (c) of a σ-algebra: $X \setminus A_j \in \mathcal{A}$ and $\cup_{j=1}^{\infty}(X \setminus A_j) \in \mathcal{A}$. Applying again property (b) we conclude that $\cap_{j=1}^{\infty} A_j = X \setminus \big(\cup_{j=1}^{\infty} (X \setminus A_j)\big) \in \mathcal{A}$.

3) We have that $A \setminus B = A \cap (X \setminus B) \in \mathcal{A}$ by property (b) and part 2).

4) As $A \subseteq B$ we have that $B = A \cup (B \setminus A)$ and this union is disjoint. Hence, by property b) of a measure we conclude that $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$. Now, if $\mu(A) < \infty$ we can substract it from both members and so $\mu(B \setminus A) = \mu(B) - \mu(A)$.

5) Let $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, ..., $B_n = A_n \setminus (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$. Then, it is easy to check that the collection $\{B_j\}$ is disjoint, $A_j \subseteq B_j$ and $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} B_j$. Hence, by property b) of a measure and part 4)

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j) \leq \sum_{j=1}^{\infty} \mu(A_j)\,.$$

Problem 2 (3 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow \mathbb{R}$ be an integrable function.

a) Prove Markov's inequality:

$$\mu(\{x \in X : \ |f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X |f|\, d\mu\,.$$

b) Using Markov's inequality, show that if f is a measurable function, then

$$\begin{array}{llll} \text{b1)} & \int |f|\, d\mu \ = 0 & \Longleftrightarrow & \mu(f \neq 0) = 0\,, \\ \text{b2)} & \int |f|\, d\mu \ < \infty & \Longrightarrow & \mu(|f| = \infty) = 0\,. \end{array}$$

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a) $\mu(\{x \in X : \ |f(x)| \geq \varepsilon\}) = \displaystyle\int_{\{|f| \geq \varepsilon\}} 1\, d\mu \leq \int_{\{|f| \geq \varepsilon\}} \frac{1}{\varepsilon} |f|\, d\mu \leq \frac{1}{\varepsilon} \int_X |f|\, d\mu.$

b1) $(\Leftarrow)$ $\displaystyle\int_X |f|\, d\mu = \int_{\{|f|=0\}} |f|\, d\mu + \int_{\{|f| \neq 0\}} |f|\, d\mu = 0 + 0 = 0.$

$(\Rightarrow)$ Using part a) we have that $\mu(\{x \in X : |f| \geq 1/n\}) = 0$ for all $n \in \mathbb{N}$, and so

$$\mu(\{x \in X : f(x) \neq 0\}) = \mu\Big(\bigcup_{n=1}^{\infty} \{x \in X : |f(x)| \geq 1/n\}\Big) \leq \sum_{n=1}^{\infty} \mu(\{x \in X : |f| \geq 1/n\}) = 0\,.$$

b2) Using part a) we have that for all $n \in \mathbb{N}$, and since $\int_X |f|\, d\mu < \infty$:

$$\mu(\{x \in X : f(x) = \infty\}) \leq \mu(\{x \in X : |f| \geq n\}) \leq \frac{1}{n} \int_X |f|\, d\mu \to 0\,, \qquad \text{as } n \to \infty\,.$$

Hence, $\mu(\{x \in X : f(x) = \infty\}) = 0$.

Problem 3 (3 points) Prove that $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1$

Let $g_n(x) = (1 + \frac{x}{2})^n e^{-2x} \chi_{[0,n]}(x)$. As $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$, we have $\displaystyle\lim_{n\to\infty} g_n(x) = e^{-x}$. Hence, we guess that:

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = \lim_{n\to\infty} \int_0^{\infty} g_n(x)\, dx = \int_0^{\infty} \Big(\lim_{n\to\infty} g_n(x)\Big)\, dx = \int_0^{\infty} e^{-x}\, dx = 1.$$

To prove it, we will show that $|g_n(x)| \leq e^{-x} \in L^1(0,\infty)$ and then our conjecture will be a consequence of the dominated convergence theorem. To do that it is enough to prove that $(1+\frac{x}{n})^n \leq e^x$ if $x \in [0,n]$. This inequality is equivalent to $n\log(1+\frac{x}{n}) \leq x$. If we define $G(x) := x - n\log(1+\frac{x}{n})$ for $x \in [0,n]$, then we must prove that $G(x) \geq 0$ for $x \in [0,n]$. But

$$G'(x) = 1 - \frac{1}{1+\frac{x}{n}} = \frac{x/n}{1+\frac{x}{n}} \geq 0 \implies G \text{ is increasing} \implies G(x) \geq G(0) = 0\,.$$
