uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

INTEGRATION AND MEASURE | Intermediate Control 2

Time: **100 minutes**

Problem 1 (2,5 points)

a) **(1 point)** Let $X = \{a, b, c, d\}$. Construct the σ-algebra generated by $\mathcal{E} = \{\{a\}\}$ and $\mathcal{E} = \{\{a\}, \{b\}\}$.

b) **(1,5 points)** Let $E \in \mathcal{A}$ be a fixed measurable subset of X. We define $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{A}$. Using that μ is a measure in X, prove that μ_E is also a measure in X.

Problem 2 (2,5 points)

a) (1 point) State the monotone convergence theorem.

b) (1.5 points) Prove that the function $f(x) = \frac{1}{\sqrt{x}}$ if $x \in (0, 1]$, and $f(0) = 0$, is Lebesgue-integrable in $[0, 1]$ and calculate its integral.

Problem 3 (3 points)

a) (1 point) State the dominated convergence theorem.

b) (2 points) Using this last theorem, compute the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx \,.$$

Problem 4 (2 points) Let us consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, with μ the counting measure and the product measure space $(\mathbb{N} \times \mathbb{N}, \mathcal{P}(\mathbb{N} \times \mathbb{N}), \mu \otimes \mu)$. Let us define the function

$$g(m, n) = \begin{cases} 1 + 2^{-m} & \text{if} \quad m = n\,, \\ -1 - 2^{-m} & \text{if} \quad m = n + 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Check that $\int_{\mathbb{N}} (\int_{\mathbb{N}} f(m, n)\, d\mu(m))\, d\mu(n)$, and $\int_{\mathbb{N}} (\int_{\mathbb{N}} f(m, n)\, d\nu(n))\, d\mu(m))$ exist and are distinct and that $\int_{\mathbb{N}\times\mathbb{N}} |f(m, n)|\, d(\mu \otimes \mu)(m, n) = \infty$. What is the relevance of this result?