Problem 1 (2,5 points) Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f, g : X \longrightarrow [0, \infty]$ be measurable positive functions, $E \in \mathcal{A}$ and $\lambda \geq 0$.

a) Define the integral of f. By using this definition:

b) Prove that $\int_E \lambda f \, d\mu = \lambda \int_E f \, d\mu$.

c) Prove that $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

d) Prove that $f \leq g \implies \int_E f \, d\mu \leq \int_E g \, d\mu$.

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a) If $s = \sum_{j=1}^n c_j \chi_{A_j}$ is a simple function with $c_j \in \mathbb{R}$ and $A_j \in \mathcal{A}$, then we define

$$\int_X s \, d\mu = \sum_{j=1}^n c_j \, \mu(A_j)$$

and if $E \in \mathcal{A}$:

$$\int_E s \, d\mu = \sum_{j=1}^n c_j \, \mu(A_j \cap E) \, .$$

Now if f is a positive function we define

$$\int_E f \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_E s \, d\mu \, .$$

b) Let us observe that s is a simple function if and only if $\tilde{s} = s/\lambda$ is also a simple function. Hence

$$\int_E \lambda f \, d\mu = \sup_{\substack{0 \leq s \leq \lambda f \\ s \text{ simple}}} \int_E s \, d\mu = \sup_{\substack{0 \leq s/\lambda \leq f \\ s \text{ simple}}} \lambda \int_E \frac{s}{\lambda} \, d\mu = \lambda \sup_{\substack{0 \leq \tilde{s} \leq f \\ \tilde{s} \text{ simple}}} \int_E \tilde{s} \, d\mu = \lambda \int_E f \, d\mu \, .$$

c) We have that

$$\int_E f \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_E s \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s = \sum_j c_j \chi_{A_j}}} \sum_j c_j \, \mu(A_j \cap E) \, .$$

Now, it is easy to check that

$$\int_X f \, \chi_E \, d\mu = \sup_{\substack{0 \leq \tilde{s} \leq f \, \chi_E \\ \tilde{s} = \sum_j c_j \chi_{A_j \cap E}}} \int_X \tilde{s} \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s = \sum_j c_j \chi_{A_j}}} \sum_j c_j \, \mu(A_j \cap E) = \int_E f \, d\mu \, .$$

d) As $f \leq g$ we have that

$$\{s \text{ simple} : \; 0 \leq s \leq f\} \subseteq \{s \text{ simple} : \; 0 \leq s \leq g\}$$

and so

$$\int_E f\,d\mu = \sup_{0\le s\le f}\int_E s\,d\mu \le \sup_{0\le s\le g}\int_E s\,d\mu = \int_E g\,d\mu\,.$$

Problem 2 (2,5 points) Consider $a > 0$.

a) Prove that for each $x \ge a$ the function

$$v(t) = \frac{t}{1+t^2x^2}$$

decreases for $t \ge 1/a$.

b) Find an upper bound of

$$\frac{n}{1+n^2x^2}$$

for every $x \ge a$ and $n \ge 1/a$, by a function which just depends on x and a.

c) Calculate

$$L = \lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\,dx\,,$$

and say what theorem you used.

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a) We have that

$$v'(t) = \frac{1-t^2x^2}{(1+t^2x^2)^2} = 0 \qquad \Leftrightarrow \qquad t^2 = \frac{1}{x^2}$$

and therefore, since $x \ge a > 0$,

$$t \ge \frac{1}{a} \implies t \ge \frac{1}{x} \implies t^2 \ge \frac{1}{x^2} \implies 1 - x^2t^2 \le 0 \implies v'(t) \le 0\,.$$

Hence $v(t)$ decreases in the interval $[1/a,\infty)$.

b) As a consequence of a)

$$v(t) \le v(1/a) = \frac{a}{a^2+x^2} \qquad \text{if } t \ge 1/a\,.$$

Therefore, if $n \ge 1/a$,

$$\frac{n}{1+n^2x^2} = v(n) \le v(1/a) = \frac{a}{a^2+x^2}\,.$$

c) As $F(x) = \dfrac{a}{a^2+x^2} \in L^1(a,\infty)$, by the dominated convergence theorem:

$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\,dx = \int_a^\infty \lim_{n\to\infty}\frac{n}{1+n^2x^2}\,dx = \int_a^\infty 0\,dx = 0\,.$$

Problem 3 (2,5 points) Consider $a \in \mathbb{R}$.

a) Explain why we can derive the parametric integral $G(a) = \displaystyle\int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right) dx$ when $a \neq 0$.

b) Obtain explicitly $G(a)$ by deriving with respect to the parameter and integrating later with respect to it. You can use, without a proof, that G is a continuous function on $\mathbb{R}$.

Hint: Since G is a continuous even function, it suffices to consider the case $a > 0$; if we consider two constants $0 < \varepsilon < M$ and $a \in [\varepsilon, M]$, find a bound of $\left|\frac{\partial}{\partial a}\left[\log(1+\frac{a^2}{x^2})\right]\right|$ by a function (which just depends on x, ε and M) in $L^1(0,\infty)$.

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a) As

$$\frac{\partial}{\partial a}\left[\log\left(1 + \frac{a^2}{x^2}\right)\right] = \frac{2a}{x^2 + a^2} \leq \frac{2M}{x^2 + \varepsilon^2} \in L^1(0,\infty)$$

for all $a \in [\varepsilon, M]$ with $0 < \varepsilon < M < \infty$, using the theorem on differentiation of parametric integrals we deduce that $G(a)$ is derivable in (ε, M) for all ε and M and therefore, since G is also even, is derivable in $\mathbb{R} \setminus \{0\}$ and

$$G'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[\log\left(1 + \frac{a^2}{x^2}\right)\right] dx = \int_0^\infty \frac{2a}{x^2 + a^2}\, dx\,.$$

b) Therefore

$$G'(a) = \int_0^\infty \frac{2a}{x^2 + a^2}\, dx = 2\left[\arctan\frac{x}{a}\right]_{x=0}^{x=\infty} = 2\,\frac{\pi}{2} = \pi\,.$$

This implies that $G(a) = \pi a + c$ for $a > 0$, where c is a constant. As G is continuous in $\mathbb{R}$, we deduce that $G(0) = c$. But, it is clear from the definition of G that $G(0) = 0$. Hence, $G(a) = \pi a$ for $a \geq 0$. Since $G(a)$ is an even function, we conclude that $G(a) = \pi|a|$ for $a \in \mathbb{R}$.

Problem 4 (2,5 points) Find a solution of the initial value problem for the heat equation:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = e^{-x^2}, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Hint: The solution of the differential equation $\frac{d}{dt}y(t) = ay(t)$ is $y(t) = Ce^{at}$, where C is a constant (it does not depend on t).

Applying the Fourier transform to the PDE equation we obtain:

$$u_t = u_{xx} \implies \hat{u}_t = -\omega^2 \hat{u}\,.$$

For each fixed $w \in \mathbb{R}$ this last equation is an ordinary differential equation whose solution is $\hat{u}(\omega,t) = C(\omega)\, e^{-\omega^2 t}$ where $\omega \in \mathbb{R}$ and $t > 0$. From this equation we deduce that $C(\omega) = \hat{u}(\omega, 0)$,

But using now the initial condition $u(x,0) = e^{-x^2}$ we deduce taking again Fourier transforms that $C(\omega) = \hat{u}(\omega,0) = \mathcal{F}[e^{-x^2}](\omega) = \frac{1}{\sqrt{4\pi}}\, e^{-\omega^2/4}$.
Hence,

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{4\pi}}\, e^{-\omega^2/4} e^{-\omega^2 t} = \frac{1}{\sqrt{4\pi}}\, e^{-\omega^2(t+1/4)}$$

and taking now the inverse Fourier transform, we obtain that

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \mathcal{F}^{-1}[e^{-\omega^2(t+1/4)}](x) = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{\pi}{t+1/4}}\, e^{-x^2/[4(t+1/4)]} = \frac{1}{\sqrt{4t+1}}\, e^{-x^2/(4t+1)}\,.$$