uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

INTEGRATION AND MEASURE | EVALUATION TEST 1

Time: **3 hours**

Problem 1 (2,5 points) Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f, g : X \longrightarrow [0, \infty]$ be measurable positive functions, $E \in \mathcal{A}$ and $\lambda \geq 0$.

a) Define the integral of f. By using this definition:

b) Prove that $\int_E \lambda f \, d\mu = \lambda \int_E f \, d\mu$.

c) Prove that $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

d) Prove that $f \leq g \implies \int_E f \, d\mu \leq \int_E g \, d\mu$.

Problem 2 (2,5 points) Consider $a > 0$.

a) Prove that for each $x \geq a$ the function

$$v(t) = \frac{t}{1 + t^2 x^2}$$

decreases for $t \geq 1/a$.

b) Find an upper bound of

$$\frac{n}{1 + n^2 x^2}$$

for every $x \geq a$ and $n \geq 1/a$, by a function which just depends on x and a.

c) Calculate

$$L = \lim_{n \to \infty} \int_a^\infty \frac{n}{1 + n^2 x^2} \, dx \,,$$

and say what theorem you used.

Problem 3 (2,5 points) Consider $a \in \mathbb{R}$.

a) Explain why we can derive the parametric integral $G(a) = \displaystyle\int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right) dx$ when $a \neq 0$.

b) Obtain explicitly $G(a)$ by deriving with respect to the parameter and integrating later with respect to it. You can use, without a proof, that G is a continuous function on $\mathbb{R}$.

Hint: Since G is a continuous even function, it suffices to consider the case $a > 0$; if we consider two constants $0 < \varepsilon < M$ and $a \in [\varepsilon, M]$, find a bound of $\left|\frac{\partial}{\partial a}\left[\log(1+\frac{a^2}{x^2})\right]\right|$ by a function (which just depends on x, ε and M) in $L^1(0,\infty)$.

Problem 4 (2,5 points) Find a solution of the initial value problem for the heat equation:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = e^{-x^2}, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Hint: The solution of the differential equation $\frac{d}{dt}y(t) = ay(t)$ is $y(t) = Ce^{at}$, where C is a constant (it does not depend on t).

You can use the following properties and formulas for the Fourier transform:

$$\mathcal{F}\left[e^{-\alpha x^2}\right](\omega) = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-\omega^2/(4\alpha)},$$

$$\mathcal{F}\left[\sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/(4\alpha)}\right](\omega) = e^{-\alpha\omega^2},$$

$$\mathcal{F}\left[f''(x)\right](\omega) = -\omega^2\mathcal{F}\left[f(x)\right](\omega)\,.$$