**Problem 1 (2,5 points)** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of measurable functions such that

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty \, .$$

Prove that:

a) The series $\sum_n f_n$ converges almost everywhere in $X$ to a function $f : X \longrightarrow \mathbb{R}$:

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \, , \qquad \text{for almost every } x \in X \, .$$

b) $f \in L^1(\mu)$.

c) $\displaystyle\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu \, .$

*Hints:* a) Consider the function $F(x) := \sum_{n=1}^{\infty} |f_n(x)|$ and show that it belongs to $L^1(X)$. c) $g_n := f_1 + \cdots + f_n$ verifies $\lim_{n\to\infty} g_n(x) = f(x)$ a.e. and $|g_n| \le F$.

---

a) Let $F(x) = \sum_{n=1}^{\infty} |f_n(x)| \in [0, \infty]$. Then, as a consequence of monotone convergence theorem

$$\int_X F \, d\mu = \int_X \lim_{N\to\infty} \sum_{n=1}^{N} |f_n(x)| \, d\mu(x) = \lim_{N\to\infty} \int_X \sum_{n=1}^{N} |f_n(x)| \, d\mu(x)$$
$$= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X |f_n(x)| \, d\mu(x) = \sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty \, ,$$

by hypothesis. Therefore:

$$F \in L^1(\mu) \implies F(x) < \infty \text{ a.e.} \implies \sum_{n=1}^{\infty} |f_n(x)| < \infty \text{ a.e.} \implies \sum_{n=1}^{\infty} f_n(x) \text{ converges a.e.}$$

b) As $|f(x)| = \left| \sum_{n=1}^{\infty} f_n(x) \right| \le \sum_{n=1}^{\infty} |f_n(x)| = F(x) \in L^1(\mu)$ we conclude that also $f \in L^1(\mu)$.

c) Let $s_N(x) = \sum_{n=1}^{N} f_n(x)$. Then:

$$|s_N(x)| \le \sum_{n=1}^{N} |f_n(x)| \le F(x) \in L^1(\mu) \qquad \text{and} \qquad s_N(x) \to f(x) \quad \text{as } N \to \infty \, ,$$

and so, by the dominated convergence theorem:

$$\int_X f \, d\mu = \int_X \lim_{N\to\infty} s_N(x) \, d\mu(x) = \lim_{N\to\infty} \int_X s_N(x) \, d\mu(x) = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n \, d\mu = \sum_{n=1}^{\infty} \int_X |f_n| \, d\mu \, .$$

---

**Problem 2 (2,5 points)**

**a)** Prove that the sequence of functions $\{f_n\}$ defined as follows

$$f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$$

is decreasing on $n$ for every $x \geq 0$.

**b)** Calculate

$$L = \lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\, dx\,,$$

and say what theorem you used.

---

a) We have that

$$f_n(x) \geq f_{n+1}(x) \Leftrightarrow (1+nx^2)(1+x^2) \geq 1+(n+1)x^2 \Leftrightarrow nx^4 \geq 0$$

and this is obviously true.

b) First, observe that

$$\begin{aligned}\int_0^\infty f_2(x)\, dx &= \int_0^\infty \frac{1+2x^2}{(1+x^2)^2}\, dx \leq \int_0^1 (1+2x^2)\, dx + \int_1^\infty \frac{1+2x^2}{x^4}\, dx \\ &= \int_0^1 (1+2x^2)\, dx + \int_1^\infty \frac{1}{x^4}\, dx + 2\int_1^\infty \frac{1}{x^2}\, dx < \infty\,.\end{aligned}$$

Hence, using the monotone convergence theorem for decreasing sequences:

$$L = \lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty \lim_{n\to\infty} f_n(x)\, dx = \int_0^\infty 0\, dx = 0,$$

since, for $x \neq 0$,

$$0 \leq \lim_{n\to\infty} f_n(x) \leq \lim_{n\to\infty} \frac{1+nx^2}{1+nx^2+\frac{n(n-1)}{2}x^4} = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{x^2}{n}}{\frac{1}{n^2}+\frac{x^2}{n}+\frac{n-1}{2n}x^4} = \frac{0+0}{0+0+\frac{x^4}{2}} = 0\,.$$

---

**Problem 3 (2,5 points)** Consider $p > -1$.

**a)** Explain why we can derive the parametric integral $H(p) = \displaystyle\int_0^1 \frac{x^p-1}{\log x}\, dx\,.$

**b)** Obtain explicitly $H(p)$ deriving with respect to the parameter and integrating later with respect to it.

a) As $p > -1$ we have that

$$\left|\frac{\partial}{\partial p}\left[\frac{x^p - 1}{\log x}\right]\right| = x^p \in L^1(0,1)$$

and so, by the theorem on derivation of parametric integrals, we conclude that $H(p)$ is derivable in $(-1, \infty)$.

b) Also, this same theorem gives that

$$H'(p) = \int_0^1 \frac{\partial}{\partial p}\left[\frac{x^p - 1}{\log x}\right] dx = \int_0^1 x^p \, dx = \left[\frac{x^{p+1}}{p+1}\right]_{x=0}^{x=1} = \frac{1}{p+1},$$

and therefore

$$H(p) = \int \frac{1}{p+1} \, dp = \log(p+1) + c.$$

As $H(0) = \int_0^1 0 \, dx = 0$ we obtain that $0 = H(0) = \log 1 + c = 0 + c = c$ and therefore $H(p) = \log(p+1)$.

---

**Problem 4 (2,5 points)** Solve, for $\omega \neq \omega_0$, the initial value problem

$$\begin{cases} x'' + \omega_0^2 \, x = k \, \sin \omega t, & \text{if } t > 0, \\ x(0) = x'(0) = 0. \end{cases}$$

*Hints*: You can use the following properties and formulas for the Laplace transform:

$$\mathcal{L}[f''](s) = s^2 \mathcal{L}[f](s) - s f(0) - f'(0),$$
$$\mathcal{L}[\sin at](s) = \frac{a}{s^2 + a^2} \qquad (s > 0).$$

---

Applying the Laplace transform to the equation we obtain:

$$L[x''](s) + \omega_0^2 \, L[x](s) = k \, L[\sin \omega t](s) \implies s^2 L[x](s) - s\, x(0) - x'(0) + \omega_0^2 \, L[x](s) = \frac{k\omega}{s^2 + \omega^2}$$

and therefore

$$L[x](s) = \frac{k\omega}{(s^2 + \omega^2)(s^2 + \omega_0^2)}.$$

Decomposing this fraction into simple fractions we obtain:

$$L[x](s) = \frac{k\omega}{(s^2 + \omega^2)(s^2 + \omega_0^2)} = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + \omega_0^2}$$

with

$$A = C = 0 \qquad \text{and} \qquad B = -D = \frac{k\omega}{\omega_0^2 - \omega^2}.$$

Therefore:

$$L[x] = \frac{B}{s^2 + \omega^2} + \frac{D}{s^2 + \omega_0^2} \implies x(t) = \frac{B}{\omega} \sin \omega t + \frac{D}{\omega_0} \sin \omega_0 t = \frac{k}{\omega_0} \, \frac{\omega_0 \sin \omega t - \omega \sin \omega_0 t}{\omega_0^2 - \omega^2}.$$