uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

INTEGRATION AND MEASURE | EVALUATION TEST 2

Time: **3 hours**

Problem 1 (2,5 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of measurable functions such that

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty \, .$$

Prove that:

a) The series $\sum_n f_n$ converges almost everywhere in X to a function $f : X \longrightarrow \mathbb{R}$:

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \, , \qquad \text{for almost every } x \in X \, .$$

b) $f \in L^1(\mu)$.

c) $\displaystyle\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu \, .$

Hints: a) Consider the function $F(x) := \sum_{n=1}^{\infty} |f_n(x)|$ and show that it belongs to $L^1(X)$.
c) $g_n := f_1 + \cdots + f_n$ verifies $\lim_{n\to\infty} g_n(x) = f(x)$ a.e. and $|g_n| \leq F$.

Problem 2 (2,5 points)

a) Prove that the sequence of functions $\{f_n\}$ defined as follows

$$f_n(x) = \frac{1 + nx^2}{(1 + x^2)^n}$$

is decreasing on n for every $x \geq 0$.

b) Calculate

$$L = \lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\, dx\,,$$

and say what theorem you used.

Problem 3 (2,5 points) Consider $p > -1$.

a) Explain why we can derive the parametric integral $H(p) = \displaystyle\int_0^1 \frac{x^p - 1}{\log x}\, dx\,.$

b) Obtain explicitly $H(p)$ deriving with respect to the parameter and integrating later with respect to it.

Problem 4 (2,5 points) Solve, for $\omega \neq \omega_0$, the initial value problem

$$\begin{cases} x'' + \omega_0^2\, x = k\, \sin \omega t\,, & \text{if } t > 0\,, \\ x(0) = x'(0) = 0\,. \end{cases}$$

Hints: You can use the following properties and formulas for the Laplace transform:

$$\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - s f(0) - f'(0)\,,$$
$$\mathcal{L}[\sin at](s) = \frac{a}{s^2+a^2} \qquad\qquad (s > 0)\,.$$