INTEGRATION AND MEASURE | TEST 3 - SOLUTIONS

Problem 1 (2,5 points)

a) What type of measure is a Borel-Stieltjes measure? What is a distribution function?

b) Let us consider the function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ x & \text{if} \quad 1 \leq x < 3 \\ 4 & \text{if} \quad x \geq 3 \end{cases}$$

Let μ_F be the Borel-Stieltjes measure with distribution function F. Calculate:

$$\mu_F(\{1\}), \quad \mu_F(\{2\}), \quad \mu_F((1,3]), \quad \mu_F((1,3)), \quad \mu_F([1,3]),.$$

c) Give an example of a distribution function F such that

$$\mu_F((a,b)) < F(b) - F(a) < \mu_F([a,b]) , \qquad \text{for some } a \text{ and } b.$$

Solution: a) μ is a Borel-Stieltjes measure is μ is a Radon measure (the compact sets have finite measure) in $\mathbb{R}$ such that

$$\mu([a,b)) = g(b^-) - g(a^-) , \qquad \forall a, b \in \mathbb{R},$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing function. This function is known as the distribution function of μ and we write $\mu = \mu_g$.

b) We have:

$$\begin{aligned}
\mu_F(\{1\}) &= F(1^+) - F(1^-) = \lim_{x\to 1^+} x - \lim_{x\to 1^-} 0 = 1 - 0 = 1 , \\
\mu_F(\{2\}) &= F(2^+) - F(2^-) = \lim_{x\to 2^+} x - \lim_{x\to 2^-} x = 0 - 0 = 0 , \\
\mu_F((1,3]) &= F(3^+) - F(1^+) = \lim_{x\to 3^+} 4 - \lim_{x\to 1^+} x = 4 - 1 = 3 , \\
\mu_F((1,3)) &= F(3^-) - F(1^+) = \lim_{x\to 3^-} x - \lim_{x\to 1^+} x = 3 - 1 = 2 , \\
\mu_F([1,3]) &= F(3^+) - F(1^-) = \lim_{x\to 3^-} 4 - \lim_{x\to 1^-} 0 = 4 - 0 = 4 ,
\end{aligned}$$

c) It holds for the function F above and $a = 1$, $b = 3$:

$$2 = \mu_F((1,3)) < F(3) - F(1) = 3 < 4 = \mu_F([1,3]) .$$

Problem 2 (2,5 points)

a) Prove that the sequence of functions

$$f_n(t) = \left(1 + \frac{t}{n}\right)^n , \qquad t \geq 0 ,$$

verify that $f_3(t) \leq f_n(t)$ for $n \geq 3$.

b) Calculate

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx\,.$$

State correctly the results and theorems you need to arrive at the solution.

Hint for b): To start, do the change of variable $t = nx$.

Solution: a) We have that

$$\left(1+\frac{t}{3}\right)^3 \le \left(1+\frac{t}{n}\right)^n \qquad \Leftrightarrow \qquad 3\,\log\left(1+\frac{t}{3}\right) \le n\log\left(1+\frac{t}{n}\right)$$

and if we define $F(t) = n\log\left(1+\frac{t}{n}\right) - 3\,\log\left(1+\frac{t}{3}\right)$ we have

$$F'(t) = \frac{t/3 - t/n}{(1+t/n)(1+3/n)} \ge 0 \implies \text{F is increasing.}$$

Hence $F(t) \ge F(0) = 0$.

b) Doing the change of variable $t = nx$ we obtain that:

$$\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \int_0^n \frac{1+t}{(1+\frac{t}{n})^n}\,dt \le \int_0^n \frac{1+t}{(1+\frac{t}{3})^3}\,dt$$

using the part a) we obtain for $n \ge 3$

$$\frac{1+t}{(1+\frac{t}{n})^n}\,\chi_{[0,n]}(t) \le \frac{1+t}{(1+\frac{t}{3})^3}\,\chi_{[0,\infty)}(t) \in L^1[0,\infty)$$

since

$$\int_0^\infty \frac{1+t}{(1+\frac{t}{3})^3}\,dt \le \int_0^1 (1+t)\,dt + \int_1^\infty \frac{1+t}{\frac{t^3}{27}}\,dt = \int_0^1 (1+t)\,dt + 27\int_1^\infty \frac{dt}{t^3} + 27\int_1^\infty \frac{dt}{t^2} < \infty\,.$$

Hence, since

$$\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n = e^t\,, \qquad t \ge 0\,,$$

using the dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \int_0^\infty \lim_{n\to\infty}\frac{1+t}{(1+\frac{t}{n})^n}\,dt = \int_0^\infty (1+t)\,e^{-t}\,dt\,.$$

Now, the sequence of positive functions $G_N(t) = (1+t)\,e^{-t}\,\chi_{[0,N]}(t)$ is clearly increasing, and so, by the monotone convergence theorem:

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \int_0^1 \lim_{N\to\infty} G_N(t)\,dt = \lim_{N\to\infty}\int_0^N (1+t)\,e^{-t}\,dt\,.$$

But $(1+t)e^{-t}$ is continuous in $[0,N]$ for all N and therefore is Riemann-integrable in $[0,N]$. Hence we can use Barrow's rule. By using integration by parts we can compute easily a primitive: $u = 1+t \implies du = dt$, $dv = e^{-t} \implies v = -e^{-t}$,

$$\int (1+t)\, e^{-t}\, dt = -(1+t)e^{-t} + \int e^{-t}\, dt = -(1+t)e^{-t} - e^{-t} = -(2+t)e^{-t}\,.$$

Finally

$$\lim_{n\to\infty} \int_0^1 \frac{n+n^2x}{(1+x)^n}\, dx = \lim_{N\to\infty} [-(2+t)e^{-t}]_{t=0}^{t=N} = 2 - \lim_{N\to\infty} \frac{2+N}{e^N} = 2 - \lim_{N\to\infty} \frac{1}{e^N} = 2 - 0 = 2\,,$$

where we have used L'Hopital rule.

Problem 3 (2,5 points)

a) State Fubini's theorem for general functions.

b) Prove that the function $f(x,y) = e^{-y}\sin 2xy$ is integrable in $A = [0,1]\times[0,\infty)$.

c) Prove that

$$\int_0^1 e^{-y}\sin 2xy\, dx = \frac{e^{-y}}{y}\, \sin^2 y\,, \qquad \int_0^\infty e^{-y}\sin 2xy\, dy = \frac{2x}{1+4x^2}\,.$$

d) Using Fubini's theorem, prove that:

$$\int_0^\infty e^{-y}\,\frac{\sin^2 y}{y}\, dy = \frac{1}{4}\,\log 5\,.$$

Solution: b) First

$$\int_0^1 e^{-y}\sin 2xy\, dx = e^{-y}\Big[-\frac{\cos(2xy)}{2y}\Big]_{x=0}^{x=1} = e^{-y}\,\frac{1-\cos 2y}{2y} = \frac{e^{-y}}{y}\,\sin^2 y\,.$$

Secondly, using Fubini's theorem for positive functions we have that

$$\int_0^1\int_0^\infty |f(x,y)|\, dx\, dy \le \int_0^1\int_0^\infty e^{-x}\, dx\, dy = \int_0^1 dy \int_0^\infty e^{-x}\, dx < \infty\,.$$

c) Integrating by parts with $u = \sin(2xy) \implies du = 2x\cos(2xy)\, dy$, $dv = e^{-y}dy \implies v = -e^{-y}$, we have

$$\int_0^\infty e^{-y}\sin 2xy\, dy = \lim_{N\to\infty}\Big([-e^{-y}\sin(2xy)]_{y=0}^{y=N} + \int_0^\infty 2xe^{-y}\cos(2xy)\, dy\Big) = 2x\int_0^\infty e^{-y}\cos(2xy)\, dy\,.$$

Using parts again: $u = \cos(2xy) \implies du = -2x\sin(2xy)\,dy$, $dv = e^{-y}dy \implies v = -e^{-y}$, we have

$$\int_0^\infty e^{-y}\sin 2xy\,dy = 2x\int_0^\infty e^{-y}\cos(2xy)\,dy = 2x\lim_{N\to\infty}\Big([-e^{-y}\cos(2xy)]_{y=0}^{y=N} - \int_0^\infty 2xe^{-y}\sin(2xy)\,dy\Big)$$
$$= 2x - 4x^2\int_0^\infty e^{-y}\sin 2xy\,dy$$

and so

$$\int_0^\infty e^{-y}\sin 2xy\,dy = \frac{2x}{1+4x^2}\,.$$

c) Using now part b) and Fubini's theorem, we have:

$$\int_0^\infty e^{-y}\frac{\sin^2 y}{y}\,dy = \int_0^\infty\int_0^1 e^{-y}\sin 2xy\,dx\,dy = \int_0^\infty e^{-y}\frac{\sin^2 y}{y}\,dy = \int_0^1\int_0^\infty e^{-y}\sin 2xy\,dy\,dx$$
$$= \int_0^1 \frac{2x}{1+4x^2}\,dx = \Big[\frac{1}{4}\log(1+4x^2)\Big]_{x=0}^{x=1} = \frac{1}{4}\log 5\,.$$

Problem 4 (2,5 points)

a) Calculate the Laplace transform of the function $f(t) = t^2\sin(at)$. Indicate what properties you are using.

b) Solve the following initial value problems, obtaining first the Laplace transform $Y(z)$ of the solution $y(t)$ and then, anti-transforming $Y(z)$:

$$\begin{cases} y'' - y' - 2y = (t^3+t^2)\,e^{2t}\,, & \text{for } t>0\,,\\ y(0)=2\,,\ y'(0)=4\,. \end{cases}$$

Solution: a) Using the property (7) and the last Laplace transform in the table, and writing $f(t) = t\sin(at)$, we have that

$$\mathcal{L}[t^2\sin(at)](s) = \mathcal{L}[tf(t)](s) = -\frac{d}{ds}[\mathcal{L}[f(t)]](s) = -\frac{d}{ds}\Big[\frac{2as}{(s^2+a^2)^2}\Big] = \frac{8as^2-2a^3}{(s^2+a^2)^3}\,.$$

b) Transforming the equation using the Laplace transform, we have:

$$s\mathcal{L}[y'] - y'(0) - s\mathcal{L}[y] + y(0) - 2\mathcal{L}[y] = \frac{3!}{(s-2)^4} + \frac{2!}{(s-2)^3}$$
$$s\big(s\mathcal{L}[y] - y(0)\big) - y'(0) - s\mathcal{L}[y] + y(0) - 2\mathcal{L}[y] = \frac{6}{(s-2)^4} + \frac{2}{(s-2)^3}$$
$$(s^2-s-2)\mathcal{L}[y] - 2s - 2 = \frac{6}{(s-2)^4} + \frac{2}{(s-2)^3}$$
$$(s+1)(s-2)\mathcal{L}[y] = \frac{6}{(s-2)^4} + \frac{2}{(s-2)^3} + 2(s+1)$$

Hence

$$\begin{aligned}\mathcal{L}[y] &= \frac{2}{s-2} + \frac{1}{s+1}\Big(\frac{6}{(s-2)^5} + \frac{2}{(s-2)^4}\Big) = \frac{2}{s-2} + \frac{6+2(s-2)}{(s+1)(s-2)^5} \\ &= \frac{2}{s-2} + \frac{2s+2}{(s+1)(s-2)^5} = \frac{2}{s-2} + \frac{2}{(s-2)^5}\end{aligned}$$

Finally, using the table of transforms we arrive at $y(t) = 2e^{2t} + 2\frac{1}{4!}t^4e^{2t} = 2e^{2t} + \frac{1}{12}t^4e^{2t}$.
