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Calculus I

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Unit 9. Primitives

9. Primitives

Differentiating is an operation that transforms an appropriate function f into another function f', which we refer to as its derivative. It makes sense to wonder about the inverse operation, i.e., given the function f' to determine f.

Definition 9.0.1 — Primitive. A function F is called a **primitive** of f if $F' = f$. We denote this operation

$$\int f(x)\,dx = F(x). \tag{9.1}$$

(Function f is called **integrand.**)

The first question we can ask is whether the primitive, if it exists, is unique. According to Corollary 7.3.4 the answer is no —but almost so. The reason is that if F and G are such that $F' = G' = f$ (i.e., F and G are two primitives of f), then $F(x) = G(x) + c$ for some constant c. Thus, primitives are unique up to an additive constant.

Some properties of primitives are inherited from those of derivatives. For instance, primitives are *linear,* i.e., given functions f and g and constants $a, b \in \mathbb{R}$,

$$\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx. \tag{9.2}$$

We can obtain a few elementary primitives by reversing the derivatives Table 7.1. The list is shown in Table 9.1.

Some primitives have the pattern

$$\int f'\big(g(x)\big)g'(x)\,dx = f\big(g(x)\big) + c. \tag{9.3}$$

We call this primitives *immediate.*

$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^α $(\alpha \neq -1)$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$\sin x$	$-\cos x$	$\dfrac{1}{1+x^2}$	$\arctan x$
x^{-1}	$\log\lvert x\rvert$	$\cos x$	$\sin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
e^x	e^x	$\sinh x$	$\cosh x$	$\dfrac{1}{\cos^2 x}$	$\tan x$
a^x	$\dfrac{a^x}{\log a}$	$\cosh x$	$\sinh x$	$\dfrac{1}{\cosh^2 x}$	$\tanh x$

Table 9.1: Primitives $F(x)$ of some elementary functions $f(x)$ (up to the additive constant) as obtained by reversing Table 7.1. Here $\alpha \in \mathbb{R}$, $a > 0$.

Here are some important special cases:

$$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + c, \qquad \int g'(x)[g(x)]^\alpha\,dx = \frac{g(x)^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1, \tag{9.4}$$

$$\int \frac{g'(x)}{1+g(x)^2}\,dx = \arctan g(x) + c, \qquad \int \frac{g'(x)}{\sqrt{1-g(x)^2}}\,dx = \arcsin g(x) + c. \tag{9.5}$$

■ **Example 9.1** The primitive

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

has nearly the form

$$\int \frac{g'(x)}{g(x)}\,dx$$

because $(\cos x)' = -\sin x$. Then

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\int \frac{(\cos x)'}{\cos x}\,dx.$$

Therefore

$$\int \tan x\,dx = -\log|\cos x| + c. \tag{9.6}$$

By a similar argument

$$\int \cot x\,dx = \log|\sin x| + c. \tag{9.7}$$

■

■ **Example 9.2** Here is a more involved example:

$$\int \sec x\,dx = \int \frac{dx}{\cos x}.$$

In order to find this primitive, let us first compute

$$(\sec x)' = \frac{\sin x}{\cos^2 x} = \sec x \tan x, \qquad (\tan x)' = \sec^2 x.$$

Adding up these two equations we get

$$(\sec x + \tan x)' = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x).$$

Therefore

$$\frac{(\sec x + \tan x)'}{\sec x + \tan x} = \sec x$$

and from this we conclude

$$\int \sec x \, dx = \log|\sec x + \tan x| + c = \log\left|\frac{1+\sin x}{\cos x}\right| + c. \tag{9.8}$$

Similarly we obtain

$$\int \csc x \, dx = -\log|\csc x + \cot x| + c = \log\left|\frac{\sin x}{1+\cos x}\right| + c. \tag{9.9}$$

Notice that

$$\left(\frac{1+\sin x}{\cos x}\right)^2 = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{1+\sin x}{1-\sin x} = \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= \left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)^2 = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^2 = \tan^2\left(\frac{x}{2} + \frac{\pi}{4}\right),$$
$$\left(\frac{\sin x}{1+\cos x}\right)^2 = \frac{1-\cos^2 x}{(1+\cos x)^2} = \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2},$$

therefore, we have the alternative expressions

$$\log\left|\frac{1+\sin x}{\cos x}\right| = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right) = \log\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|,$$
$$\log\left|\frac{\sin x}{1+\cos x}\right| = \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right) = \log\left|\tan\frac{x}{2}\right|$$

■

9.1 Integration by parts

Theorem 9.1.1 If f and g are two differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{9.10}$$

Proof. Since $f(x)g(x)$ is the primitive of $\left[f(x)g(x)\right]'$, we have

$$f(x)g(x) = \int \left[f(x)g(x)\right]' dx = \int \left[f'(x)g(x) + f(x)g'(x)\right] dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

From here equation (9.10) follows straight away. ■

This is one of the most useful integration techniques, as a few examples will reveal.

■ **Example 9.3** A classic example is the integral

$$\int xe^x \, dx.$$

Since $e^x = (e^x)'$ it is easy to recognise the left-hand side of (9.10). Therefore

$$\int xe^x \, dx = xe^x - \int 1 \cdot e^x \, dx = xe^x - e^x + c = (x-1)e^x + c.$$

■

This example can be generalised whenever we have a function $g'(x)$ easy to integrate several times (e.g., an exponential, a power, a trigonometric function...) multiplied by a polynomial. The polynomial plays the role of function $f(x)$, and we have to apply integration by parts as many times as the degree of the polynomial. (The example above is one of those cases, in which the polynomial has degree 1.)

Exercise 9.1 Calculate

$$\int (x^2+1)\sin(2x-1)\, dx.$$

■

■ **Example 9.4** Often we cannot see $g'(x)$ explicitly because $g'(x) = 1$. For example, in the integral

$$\int \log x \, dx.$$

If $g'(x) = 1$ then $g(x) = x$, therefore

$$\int \log x \, dx = x\log x - \int x \cdot \frac{1}{x} \, dx = x\log x - \int dx = x\log x - x + c.$$

Thus, we can add the primitive of yet another elementary function to our list:

$$\int \log x \, dx = x\log x - x + c. \tag{9.11}$$

■

We can generalise this example to obtain the primitive of an inverse f^{-1} if we know that $F(x)$ is a primitive of $f(x)$:

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int x\left(f^{-1}\right)'(x)\, dx.$$

But $x = f\left(f^{-1}(x)\right)$, therefore

$$\int x(f^{-1})'(x)\, dx = \int f\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x)\, dx = F\left(f^{-1}(x)\right)$$

because the last integral matches the pattern of an immediate integral. Thus, we can conclude:

Theorem 9.1.2 If function f has an inverse f^{-1} and $F'(x) = f(x)$, then

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F\left(f^{-1}(x)\right). \tag{9.12}$$

■ **Example 9.5** We know that

$$\int \tan x\,dx = -\log|\cos x| \equiv F(x)$$

is a primitive of $f(x) = \tan x$. Therefore

$$\int \arctan x\,dx = x\arctan x + \log|\cos(\arctan x)| + c.$$

We can simplify this expression if we rewrite the cosine in terms of the tangent. Since

$$\cos^2 x = \frac{1}{1+\tan^2 x} \qquad \Rightarrow \qquad \cos x = (1+\tan^2 x)^{-1/2},$$

then

$$\log|\cos(\arctan x)| = \log|(1+x^2)^{-1/2}| = -\frac{1}{2}\log(1+x^2).$$

Thus

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\log(1+x^2) + c. \tag{9.13}$$

■

Exercise 9.2 Prove that

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + c. \tag{9.14}$$

■

■ **Example 9.6** Another typical use of the integration by parts is to recover the same integral after applying the formula. Such is the case of

$$\int \frac{\log x}{x}\,dx.$$

since $1/x = (\log x)'$,

$$\int \frac{\log x}{x}\,dx = (\log x)^2 - \int \log x \frac{dx}{x}.$$

From this we conclude that

$$2\int \frac{\log x}{x}\,dx = (\log x)^2$$

thus

$$\int \frac{\log x}{x}\,dx = \frac{1}{2}(\log x)^2 + c.$$

■

■ **Example 9.7** The integrals

$$\int e^x \sin x\, dx, \qquad \int e^x \cos x\, dx,$$

are another example of the same technique, where we have to integrate by parts more than once. In the first integration we identify $g'(x) = \sin x$ and get

$$\int e^x \sin x\, dx = e^x(-\cos x) - \int (-\cos x) e^x\, dx = -e^x \cos x + \int \cos x e^x\, dx.$$

In the second integration we identify $g'(x) = \cos x$, so

$$\int \cos x e^x\, dx = e^x \sin x - \int \sin x e^x\, dx.$$

Therefore, if we denote

$$S \equiv \int e^x \sin x\, dx, \qquad C \equiv \int e^x \cos x\, dx,$$

what we have obtained are the equations

$$S = -e^x \cos x + C, \qquad C = e^x \sin x - S.$$

Solving this system we obtain

$$S = \frac{e^x}{2}(\sin x - \cos x) + c, \qquad C = \frac{e^x}{2}(\sin x + \cos x) + c.$$

■

■ **Example 9.8** Another technique associated to the integration by parts is the construction of recurrence formulas. This is illustrated by the example

$$I_n(x) = \int \frac{dx}{(1+x^2)^n},$$

whose case $n = 1$ is straightforward: $I_1(x) = \arctan x$. In order to find the recurrence we proceed as follows:

$$I_{n+1}(x) = \int \frac{dx}{(1+x^2)^{n+1}} = \int \frac{1+x^2}{(1+x^2)^{n+1}}\, dx - \int \frac{x^2}{(1+x^2)^{n+1}}\, dx = I_n(x) - \frac{1}{2} \int x \frac{2x}{(1+x^2)^{n+1}}\, dx.$$

We now integrate by parts

$$\int x \frac{2x}{(1+x^2)^{n+1}}\, dx = -\frac{x}{n} \frac{1}{(1+x^2)^n} + \frac{1}{n} \int \frac{dx}{(1+x^2)^n} = \frac{I_n(x)}{n} - \frac{x}{n(1+x^2)^n}.$$

Thus

$$I_{n+1}(x) = I_n(x) \left(1 - \frac{1}{2n}\right) + \frac{x}{2n(1+x^2)^n} = \frac{2n-1}{2n} I_n(x) + \frac{1}{2n} \frac{x}{(1+x^2)^n}.$$

For instance,

$$I_2(x) = \frac{1}{2} \arctan x + \frac{1}{2} \frac{x}{(1+x^2)},$$

$$I_3(x) = \frac{3}{4} I_2(x) + \frac{1}{4} \frac{x}{(1+x^2)^2} = \frac{3}{8} \arctan x + \frac{3}{8} \frac{x}{(1+x^2)} + \frac{1}{4} \frac{x}{(1+x^2)^2},$$

etc. ■

9.2 Primitives of rational functions

Rational functions can be integrated thanks to a partial fractions decomposition. First of all, we can focus on rational functions

$$R(x) = \frac{P(x)}{Q(x)}, \tag{9.15}$$

where the degree of $P(x)$ is smaller than the degree of $Q(x)$, and $Q(x)$ is a monic polynomial. The reason is that if this is not true, then we can divide $P(x)$ by $Q(x)$, obtain a quotient polynomial $C(x)$ and a remainder $M(x)$, so that we can write

$$R(x) = \frac{P(x)}{Q(x)} = C(x) + \frac{M(x)}{Q(x)}.$$

In the last rational fraction the degree of $M(x)$ is smaller than the degree of $Q(x)$, and the polynomial $C(x)$ can be readily integrated.

Any monic polynomial $Q(x)$ can be factored out into a series of elementary factors, i.e.,

$$Q(x) = (x-a_1)^{n_1}\cdots(x-a_r)^{n_r}\left(x^2+p_1x+q_1\right)^{m_1}\cdots\left(x^2+p_sx+q_s\right)^{m_s}. \tag{9.16}$$

Numbers $a_1, a_2, \ldots, a_r$ are real roots of the polynomial and $n_1, n_2, \ldots, n_r$ their respective multiplicities. The quadratic factors $\left(x^2+p_jx+q_j\right)^{m_j}$ are irreducible (i.e., $p_j^2 < 4q_j$) and correspond to complex roots of the polynomial. Numbers m_j are their respective multiplicities.

It turns out that the rational function (9.15) with denominator (9.16) can be expanded as

$$R(x) = \sum_{i=1}^{r}\left[\frac{A_{i1}}{x-a_i} + \cdots + \frac{A_{in_i}}{(x-a_i)^{n_i}}\right] + \sum_{j=1}^{s}\left[\frac{B_{j1}x+C_{j1}}{x^2+p_jx+q_j} + \cdots + \frac{B_{jm_j}x+C_{jm_j}}{(x^2+p_jx+q_j)^{m_j}}\right] \tag{9.17}$$

These partial fractions are easier to integrate. A few examples will illustrate the method.

■ **Example 9.9** Calculate

$$\int \frac{2x^2-4x+6}{(x-1)^3}\,dx.$$

According to the partial fractions decomposition (9.17),

$$\frac{2x^2-4x+6}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}.$$

There are several ways to find A, B, and C. For instance, we can multiply the equation above by $(x-1)^3$ and get

$$2x^2-4x+6 = A + B(x-1) + C(x-1)^2.$$

Then setting $x=1$ we obtain $A=4$. Substituting this value of A in the previous equation and simplifying yields

$$2x^2-4x+2 = B(x-1)+C(x-1)^2 \qquad \Rightarrow \qquad 2(x-1)^2 = B(x-1)+C(x-1)^2,$$

hence $B=0$ and $C=2$.

An alternative is to obtain the Taylor polynomial for $2x^2-4x+6$ in powers of $x-1$. It is $4+2(x-1)^2$. Then

$$\frac{2x^2-4x+6}{(x-1)^3} = \frac{4+(x-1)^2}{(x-1)^3} = \frac{4}{(x-1)^3} + \frac{2}{x-1}.$$

Finally

$$\int \frac{2x^2-4x+6}{(x-1)^3}\,dx = \int \frac{4}{(x-1)^3}\,dx + \int \frac{2}{x-1}\,dx = -\frac{2}{(x-1)^2} + 2\log|x-1| + c.$$

■

■ **Example 9.10** Calculate

$$\int \frac{x+2}{x(x-1)(x-2)}\,dx.$$

The partial fraction decomposition yields

$$\frac{x+2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Multiplying by $x(x-1)(x-2)$ leads to

$$x+2 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1).$$

Setting $x=0$ we get $2=2A$, i.e., $A=1$. Setting $x=1$ we get $3=-B$, i.e., $B=-3$. Finally, setting $x=2$ we get $4=2C$, i.e., $C=2$. Thus

$$\begin{aligned}\int \frac{x+2}{x(x-1)(x-2)}\,dx &= \int \frac{dx}{x} - 3\int \frac{dx}{x-1} + 2\int \frac{dx}{x-2} \\ &= \log|x| - 3\log|x-1| + 2\log|x-2| + c.\end{aligned}$$

■

■ **Example 9.11** Calculate

$$\int \frac{x^2+1}{x^2(x-1)(x+1)}\,dx.$$

The partial fraction decomposition yields

$$\frac{x^2+1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}.$$

Multiplying by $x^2(x-1)(x+1)$ leads to

$$x^2+1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1).$$

Setting $x=0$ leads to $1=-B$, i.e., $B=-1$. Setting $x=1$ leads to $2=2C$, i.e., $C=1$. Setting $x=-1$ leads to $2=-2D$, i.e., $D=-1$. Now, substituting these constants

$$\begin{aligned}x^2+1 &= Ax(x-1)(x+1) - (x-1)(x+1) + x^2(x+1) - x^2(x-1) \\ &= Ax(x-1)(x+1) - x^2 + 1 + x^3 + x^2 - x^3 + x^2 = Ax(x-1)(x+1) + x^2 + 1,\end{aligned}$$

so $A=0$.

Now,

$$\int \frac{x^2+1}{x^2(x-1)(x+1)}\,dx = -\int \frac{dx}{x^2} + \int \frac{dx}{x-1} - \int \frac{dx}{x+1} = \frac{1}{x} + \log|x-1| - \log|x+1| + c.$$

■

■ **Example 9.12** Calculate

$$\int \frac{5x^2 - x + 3}{x(x^2+1)}\, dx.$$

The partial fraction decomposition yields

$$\frac{5x^2 - x + 3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1}.$$

Multiplying by $x(x^2+1)$leads to

$$5x^2 - x + 3 = A(x^2+1) + (Bx + C)x.$$

Setting $x = 0$ leads to $A = 3$. Substituting this value

$$5x^2 - x + 3 = 3(x^2+1) + (Bx + C)x \qquad \Rightarrow \qquad 2x^2 - x = (Bx + C)x \qquad \Rightarrow$$
$$2x - 1 = Bx + C,$$

so $B = 2$ and $C = -1$.

Then

$$\int \frac{5x^2 - x + 3}{x(x^2+1)}\, dx = 3\int \frac{dx}{x} + \int \frac{2x}{x^2+1}\, dx - \int \frac{dx}{x^2+1} = 3\log|x| + \log(x^2+1) - \arctan x + c.$$

■

■ **Example 9.13** Calculate

$$\int \frac{2x + 4}{x^2 + 2x + 2}\, dx.$$

In order to perform a partial fraction decomposition we need to find the roots of the denominator. However these roots are $-1 \pm i$, so $x^2 + 2x + 2$ is an irreducible square factor. The way to proceed in these cases is to take the first two terms and complete the square. In other words, we write $x^2 + 2x = (x+1)^2 - 1$. Thus,

$$\begin{aligned}\int \frac{2x + 4}{x^2 + 2x + 2}\, dx &= \int \frac{2(x+2)}{(x+1)^2 + 1}\, dx = \int \frac{2(x+1)}{(x+1)^2 + 1}\, dx + 2\int \frac{dx}{(x+1)^2 + 1} \\ &= \log\left[(x+1)^2 + 1\right] + 2\arctan(x+1) + c \\ &= \log(x^2 + 2x + 2) + 2\arctan(x+1) + c.\end{aligned}$$

■

■ **Example 9.14** Calculate

$$\int \frac{2x + 4}{(x^2 + 2x + 2)^2}\, dx.$$

Using the same transformation as in the previous example

$$\int \frac{2x + 4}{(x^2 + 2x + 2)^2}\, dx = \int \frac{2(x+1)}{\left[(x+1)^2 + 1\right]^2}\, dx + 2\int \frac{dx}{\left[(x+1)^2 + 1\right]^2}.$$

The first integral is immediate,

$$\int \frac{2(x+1)}{\left[(x+1)^2 + 1\right]^2}\, dx = -\frac{1}{(x+1)^2 + 1} = -\frac{1}{x^2 + 2x + 2}.$$

The second integral can be done using the recurrence derived in Example 9.8,

$$2\int \frac{dx}{[(x+1)^2+1]^2} = \arctan(x+1) + \frac{x+1}{x^2+2x+2}.$$

Thus,

$$\int \frac{2x+4}{(x^2+2x+2)^2}\,dx = \arctan(x+1) + \frac{x}{x^2+2x+2} + c.$$

■

9.3 Change of variable

Let $F(x)$ be one primitive of $f(x)$, and let $x = g(t)$ be a change from variable x to the new variable t. By the chain rule

$$\frac{d}{dt}F\big(g(t)\big) = f\big(g(t)\big)g'(t),$$

thus, integrating this equation,

$$F\big(g(t)\big) = \int f\big(g(t)\big)g'(t)\,dt.$$

But using the change $x = g(t)$ and the fact that $F(x) = \int f(x)\,dx$, we can rewrite this identity as

$$\int f(x)\,dx = \int f\big(g(t)\big)g'(t)\,dt. \tag{9.18}$$

This is the equation ruling a change of variable in the calculation of a primitive.

R A simple way to remember this rule is to rewrite dx according to

$$dx = \frac{dx}{dt}\,dt = g'(t)\,dt.$$

■ **Example 9.15** Calculate

$$\int \frac{e^x}{e^{2x}+1}\,dx.$$

Here the obvious change of variable is $e^x = t$ or $x = \log t$. Then $dx = dt/t$ and

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{t}{t^2+1}\frac{dt}{t} = \int \frac{dt}{t^2+1} = \arctan t + c = \arctan(e^x) + c.$$

■

■ **Example 9.16** Calculate

$$\int \frac{dx}{\sqrt[3]{(1-2x)^2} - \sqrt{1-2x}}.$$

Here the change of variable is $t^m = 1-2x$, choosing m so that all roots disappear. The simplest choice is the least common multiple of 2 and 3 in this case, i.e., $m=6$. So $x=(1-t^6)/2$ and therefore $dx = -3t^5\,dt$. Then,

$$\int \frac{dx}{\sqrt[3]{(1-2x)^2}-\sqrt{1-2x}} = \int \frac{-3t^5}{t^4-t^3}\,dt = -3\int \frac{t^2}{t-1}\,dt = -3\int \left(t+1+\frac{1}{t-1}\right)dt$$
$$= -\frac{3}{2}(t+1)^2 - 3\log|t-1| + c$$
$$= -\frac{3}{2}\left(1+\sqrt[6]{1-2x}\right)^2 - 3\log\left|1-\sqrt[6]{1-2x}\right| + c$$

■

■ **Example 9.17** Calculate

$$\int \frac{dx}{x\sqrt{1-x^2}}.$$

Whenever we have an expression like $\sqrt{1-x^2}$ one possible change of variable is $x=\sin t$, for then $\sqrt{1-x^2}=\cos t$ and $dx=\cos t\,dt$. In this case this leads to

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos t}{\sin t\cos t}\,dt = \int \frac{dt}{\sin t} = \log\left|\frac{\sin t}{1+\cos t}\right| + c = \log\left(\frac{|x|}{1+\sqrt{1-x^2}}\right)+c.$$

■

Suggested changes of variables:

(I) If there appear $\sqrt{1+x^2}$ then $x=\tan t$ transforms

$$\sqrt{1+x^2} = \frac{1}{\cos t}, \qquad dx = \frac{dt}{\cos^2 t},$$

or $x=\sinh t$ transforms

$$\sqrt{1+x^2} = \cosh t, \qquad dx = \cosh t\,dt.$$

(II) If there appear $\sqrt{x^2-1}$ then $x=\sec t$ transforms

$$\sqrt{x^2-1} = \tan t, \qquad dx = \sec t\tan t\,dt,$$

or $x=\cosh t$ transforms

$$\sqrt{x^2-1} = \sinh t, \qquad dx = \sinh t\,dt.$$

(III) As a last resource, in rational functions of sines and cosines we can use $t=\tan(x/2)$, which transforms

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2\,dt}{1+t^2}.$$

Problems

Problem 9.1 Obtain the following immediate (or nearly so) primitives:

(i) $\displaystyle\int \frac{dx}{\cos^2 x}$; (iv) $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$; (vii) $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$;

(ii) $\displaystyle\int \frac{\sin x-\cos x}{\sin x+\cos x}\,dx$; (v) $\displaystyle\int \frac{dx}{1-\sin x}\,dx$; (viii) $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$;

(iii) $\displaystyle\int \frac{x}{(x^2+1)^{5/2}}\,dx$; (vi) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$; (ix) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

HINTS: (iv) multiply and divide by $1-\cos x$ and expand; (v) idem with $1+\sin x$; (vii) alternatively $t=\sqrt{1-\sqrt{x}}$; (viii) $\cos^3 x=(1-\sin^2 x)\cos x$ and expand; (ix) write $x^3=x(x^2-1)+x$ and expand.

Problem 9.2 Obtain the primitives of the following rational functions:

(i) $\displaystyle\int \frac{x^2}{(x-1)^3}\,dx$; (iii) $\displaystyle\int \frac{2x^2+3}{x^2(x-1)}\,dx$; (v) $\displaystyle\int \frac{4x^4-x^3-46x^2-20x+153}{x^3-2x^2-9x+18}\,dx$;

(ii) $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$; (iv) $\displaystyle\int \frac{2}{x^2-2x+2}\,dx$; (vi) $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$.

HINTS: (ii) $x^2+x+1=(x+1/2)^2+3/4$; (v) $x^3-2x^2-9x+18=(x-2)(x-3)(x+3)$; (vi) $x^4-2x^2+1=(x-1)^2(x+1)^2$.

Problem 9.3 Obtain the following primitives doing an appropriate change of variable:

(i) $\displaystyle\int x^2\sqrt{x-1}\,dx$; (ix) $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$; (xvii) $\displaystyle\int \sqrt{\sqrt{x}+1}\,dx$;

(ii) $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$; (x) $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$; (xviii) $\displaystyle\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$;

(iii) $\displaystyle\int \cos(\log x)\,dx$; (xi) $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$; (xix) $\displaystyle\int \sqrt{2+e^x}\,dx$;

(iv) $\displaystyle\int \sin(\log x)\,dx$; (xii) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}\,dx$; (xx) $\displaystyle\int \frac{\sin x+3\cos x}{\sin x+2\cos x}\,dx$;

(v) $\displaystyle\int \cos^2(\log x)\,dx$; (xiii) $\displaystyle\int \sqrt{e^x-1}\,dx$; (xxi) $\displaystyle\int \frac{\sin x+3\cos x}{\sin x\cos x+2\sin x}\,dx$;

(vi) $\displaystyle\int \frac{\sqrt{x}+1}{x+3}\,dx$; (xiv) $\displaystyle\int \frac{\sin^2 x\cos^5 x}{\tan^3 x}\,dx$; (xxii) $\displaystyle\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$;

(vii) $\displaystyle\int \frac{(x+1)^3}{\sqrt{1-(x+1)^2}}\,dx$; (xv) $\displaystyle\int \frac{dx}{3+\sqrt{2x+5}}$; (xxiii) $\displaystyle\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$;

(viii) $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$; (xvi) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$; (xxiv) $\displaystyle\int \frac{dx}{e^x-4e^{-x}}\,dx$.

HINTS: (i) $t=\sqrt{x-1}$ (or int. by parts twice); (ii) $t^2=x^3$; (iii)–(v) $t=\log x$; (vi) $t=\sqrt{x}$; (vii) $t=\sqrt{1-(x+1)^2}$; (viii) $t=1+x^2$; (ix) $t^2=1+x$; (x) $t^3=1-x$; (xi) $t=e^x$; (xii) $t^2=e^{2x}-1$; (xiii) $t^2=e^x-1$; (xiv) $t=\cos x$; (xv) $t=3+\sqrt{2x+5}$; (xvi) $t=\sqrt{(x-1)/(x+1)}$; (xvii) $t=\sqrt{\sqrt{x}+1}$; (xviii) $t=\sqrt{x+2}$; (xix) $t=\sqrt{2+e^x}$; (xx) $t=\tan x$; (xxi) $t=\tan(x/2)$; (xxii) $t=\sqrt{1+x^{1/3}}$; (xxiii) $t^3=x+2$; (xxiv) $t=e^x$.

Problem 9.4 Obtain the following primitives with the help of some trigonometric identity:

(i) $\int \sin^2 x\,dx$; (ii) $\int \cos^2 x\,dx$; (iii) $\int \sin^4 x\,dx$; (iv) $\int \cos^4 x\,dx$; (v) $\int \cos^6 x\,dx$;

(vi) $\int \sin^2 x \cos^2 x\,dx$; (vii) $\int \tan^2 x\,dx$; (viii) $\int \tan^4 x\,dx$; (ix) $\int \frac{dx}{\cos^4 x}$; (x) $\int \sin^5 x\,dx$;

(xi) $\int \cos^3 x \sin^2 x\,dx$; (xii) $\int \sec^6 x\,dx$; (xiii) $\int \sin^3 x \cos^2 x\,dx$; (xiv) $\int \tan^3 x\,dx$; (xv) $\int \tan^3 x \sec^4 x\,dx$.

HINTS: Identities to use: $2\cos^2 x = 1+\cos 2x$; $2\sin^2 x = 1-\cos 2x$; $\cos^2 x + \sin^2 x = 1$; $\sec^2 x = 1+\tan^2 x$.

Problem 9.5 Integrate by parts to obtain the following primitives:

(i) $\int x\tan^2(2x)\,dx$; (ii) $\int e^x \sin \pi x\,dx$; (iii) $\int e^x \cos 2x\,dx$; (iv) $\int \sec^3 x\,dx$;

(v) $\int \tan^2(3x)\sec^3(3x)\,dx$; (vi) $\int e^{\sin x}\cos^3 x\,dx$; (vii) $\int x^2 \log x\,dx$; (viii) $\int x^m \log x\,dx$;

(ix) $\int (\log x)^3\,dx$; (x) $\int x(\log x)^2\,dx$; (xi) $\int \frac{x\log x}{(1+x^2)^2}\,dx$; (xii) $\int \arctan \sqrt[3]{x}\,dx$.

Problem 9.6 Obtain the following primitives by performing a trigonometric substitution:

(i) $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$; (ii) $\int \frac{x^2}{(x^2+1)^{5/2}}\,dx$;

(iii) $\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$; (iv) $\int \frac{dx}{x^2\sqrt{1-x^2}}$;

(v) $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Problem 9.7 Find recurrence formulas for the following integrals:

(i) $I_m = \int \sin^m x\,dx \longrightarrow I_m = -\frac{1}{m}\sin^{m-1} x\cos x + \frac{m-1}{m} I_{m-2}$;

(ii) $I_m = \int (\log x)^m\,dx \longrightarrow I_m = x(\log x)^m - mI_{m-1}$;

(iii) $I_m = \int x^m e^{-x}\,dx \longrightarrow I_m = -x^m e^{-x} + mI_{m-1}$;

(iv) $I_m = \int \tan^m x\,dx \longrightarrow I_m = \frac{1}{m-1}\tan^{m-1} x - I_{m-2}$;

(v) $I_m = \int \sec^m x\,dx \longrightarrow I_m = \frac{1}{m-1}\tan x \sec^{m-2} x + \frac{m-2}{m-1} I_{m-2}$;

(vi) $I_m = \int x^m e^{x^2}\,dx \longrightarrow I_m = \frac{1}{2}x^{m-1}e^{x^2} - \frac{m-1}{2} I_{m-2}$;

(vii) $I_{m,n} = \int \sin^m x \cos^n x\,dx \longrightarrow I_{m,n} = -\frac{1}{m+n}\sin^{m-1} x\cos^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}$.

Problem 9.8 Without calculating the integral, prove that

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\,dx = Ax + B\log|c\cos x + d\sin x| + \text{const.}$$

by determining the constants A and B as functions of a, b, c, and d.