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Calculus I

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Unit 1. The Real Line

1. The Real Line

In a loose sense, Calculus can be defined as the "algebra of real numbers". Real numbers are therefore its most basic ingredient. At present, we are all familiar with real numbers, but constructing them was a long process that took us more than two-thousand years. For the most intuitive numbers, those that are a human universal and that we use to count, are what mathematicians call *natural numbers*. Any other set of numbers is a construction deliberately introduced to solve a problem.

What we require of a number set so that it can be algebraically manipulated is that it satisfies a set of properties that we globally refer to as an *ordered field*. So let us start by setting up those properties and exploring what they allow us to do.

1.1 Ordered Fields

An ordered field is a set of elements —that we call *numbers*— with two binary operations: *addition* (denoted with "+") and *multiplication* (denoted with "·"), which satisfy two set of properties: *field axioms* and *order axioms*.

Definition 1.1.1 — Field. For all x, y, and z the following properties hold:

I. Addition axioms:

1. $x+y=y+x$ ***commutativity***
2. $x+(y+z)=(x+y)+z$ ***associativity***
3. There is a number 0 such that $x+0=x$ ***zero***
4. For each x there is a number, denoted $-x$, such that $x+(-x)=0$ ***inverse***

II. Multiplication axioms:

5. $x\cdot y=y\cdot x$ ***commutativity***
6. $x\cdot(y\cdot z)=(x\cdot y)\cdot z$ ***associativity***
7. There is a number 1 such that $x\cdot 1=x$ ***unity***
8. For each $x\neq 0$ there is a number, denoted x^{-1}, such that $x\cdot x^{-1}=1$ ***reciprocal***
9. $x\cdot(y+z)=x\cdot y+x\cdot z$ ***distributive law***
10. $1\neq 0$ ***nontriviality***

R Another way of representing the multiplicative reciprocal is using a division bar: $xy^{-1} = x/y = \frac{x}{y}$.

Any set with two binary opeartions satisfying these axioms is called a field. But to be an ordered field there must be also a relation, denoted "$\leqslant$" (and read "smaller than or equal to"), with the following properties:

Definition 1.1.2 For all x, y, and z the following properties hold:

III. Order axioms:

11. $x \leqslant x$ ***reflexivity***
12. If $x \leqslant y$ and $y \leqslant x$ then $x = y$ ***antisymmetry***
13. If $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$ ***transitivity***
14. Either $x \leqslant y$ or $y \leqslant x$ ***linear ordering***
15. If $x \leqslant y$ then $x + z \leqslant y + z$ ***compatibility with addition***
16. If $0 \leqslant x$ and $0 \leqslant y$ then $0 \leqslant x \cdot y$ ***compatibility with multiplication***

Properties 11–13 define an *ordering* (sometimes call *partial ordering*). Property 14 defines a *linear* or *total ordering,* meaning that any two elements can be compared and decided which one is on the left and which one on the right (alternatively, it means that we can place the elements on a straight line in such a way that $x < y$ means that x is 'to the left of' y). An ordered field is a field satisfying all properties 11–16. The last two properties are important in order to manipulate inequalities.

We will also introduce some other order symbols: "$\geqslant$", read "greater than or equal to", so that $x \geqslant y$ is equivalent to $y \leqslant x$; "$<$", read "smaller than", so that $x < y$ is equivalent to $x \leqslant y$ but $x \neq y$; "$>$", read "greater than", so that $x > y$ is equivalent to $x \geqslant y$ but $x \neq y$.

We can use properties 11, 12, and 14 to proof the following:

Proposition 1.1.1 — Law of trichotomy. For every pair of elements x and y of an ordered field, one and only one of the following relations hold: $x < y$, $x = y$, or $x > y$.

The next proposition list a set of properties that can be proven to follow from the axioms of an ordered field. You will recognise them as the standard algebraic operations we perform when manipulating equations. It is important to make clear that they strongly depend on the axioms above, hence the importance to have sets of numbers that are ordered fields in order to do algebra.

R Notice that in general we will omit the symbol "$\cdot$" in multiplications, unless it is necessary to avoid ambiguitiy.

Proposition 1.1.2 The following properties hold in any ordered field:

i. Unique neutrals If $a + x = a$ for all a, then $x = 0$. If $ax = a$ for all a, then $x = 1$.
ii. Unique inverses If $a + x = 0$, then $x = -a$. If $ax = 1$, then $x = a^{-1}$.
iii. No divisors of zero If $xy = 0$, then $x = 0$ or $y = 0$ (or both).
iv. Cancellation laws for addition If $a + x = b + x$, then $a = b$. If $a + x \leqslant b + x$, then $a \leqslant b$.
v. Cancellation laws for multiplication If $ax = bx$ and $x \neq 0$, then $a = b$. If $ax \geqslant bx$, then $a \geqslant b$ if $x > 0$ and $a \leqslant b$ if $x < 0$.
vi. $0x = 0$ for all x.
vii. $-(-x) = x$ for all x.
viii. $-x = (-1)x$ for all x.
ix. If $x \neq 0$, the $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.
x. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.
xi. If $x \leqslant y$ and $0 \leqslant z$, then $xz \leqslant yz$. If $x \leqslant y$ and $z \leqslant 0$, then $xz \geqslant yz$.
xii. If $x \leqslant 0$ and $y \leqslant 0$, then $xy \geqslant 0$. If $x \leqslant 0$ and $y \geqslant 0$, then $xy \leqslant 0$.

xiii. $0 < 1$.
xiv. $x^2 \geqslant 0$ for all x.

■ **Example 1.1** Using property **xiv** above it follows that $(a-b)^2 \geqslant 0$ for any two numbers a and b. Using twice the distributive law, $(a-b)^2 = a^2 - 2ab + b^2 \geqslant 0$. Thus, adding $2ab$ to both terms of the inequality we get $a^2 + b^2 \geqslant 2ab$. Dividing by $2(>0)$ and using property **v**,

$$ab \leqslant \frac{a^2+b^2}{2}.$$

■

■ **Example 1.2** Applying twice the distributive law we get, for any pair of numbers a and b,

$$(a-b)(a+b) = a^2 - b^2.$$

(This identity is more useful applied right to left.) ■

■ **Example 1.3** If $0 \leqslant a < b$, then it follows that $a^2 < b^2$. To see that, applying property **xi** above we obtain $a^2 \leqslant ab$ (multiplying by $a \geqslant 0$) and also $ab \leqslant b^2$ (multiplying by $b \geqslant 0$). So $a^2 \leqslant b^2$. But we can exclude that $a^2 = b^2$, for if that were true, then $b^2 - a^2 = (b-a)(b+a) = 0$. But $b + a > 0$, so the only possibility we are left with is $b = a$, which is not true. Therefore we must conclude that $a^2 < b^2$. ■

1.2 Number Systems

Let us have a look at the different number systems that have been constructed, the kind of problems that they are meant to solve and their peculiarities, up to the appearance of real numbers. There is a further set of numbers, *complex numbers,* which to a certain extent close the need for different number systems —at least from an algebraic point of view. In a way they “complete” real numbers and are necessary in linear algebra and in advanced applications of analysis. They will not be covered in this course though, so we will just mention them briefly.

1.2.1 Natural Numbers

This is the most basic set of numbers. Their meaning is intuitive and its main use is counting. All human cultures have a name for at least the first few natural numbers, and most have a way to give a name to natural numbers of arbitrary (or al least very large) size. Most cultures can add natural numbers, and many can multiply them.

In mathematics, they are introduced as the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. In more abstract terms, natural numbers are constructed with two axioms:

1. There is one (and only one) first element $1 \in \mathbb{N}$.
2. Every element $n \in \mathbb{N}$ has a *successor* $n+1$.

Notice that 1, n, or $n+1$ are just arbitrary numbers. We commonly use the arabic numerals with positional notation to name them, but their names change in different languages. Often we even denote them with roman numerals (I, II, III, IV, etc.). As a matter of fact, this is the reason why sometimes 0 is considered the first element of $\mathbb{N}$, instead of 1.

Natural numbers can be added and multiplied (multiplication is the abbreviation of a repeated addition). However, addition is conmutative and associative (and may even have a neutral element if we start in 0 rather than 1), but there is no inverse of a number. In other words, the equation $n + x = 0$ cannot be solved in $\mathbb{N}$ (there is no $x \in \mathbb{N}$ that satisfies the equation). This means that we cannot define a substraction operation that works for all pairs of natural numbers.

Multiplication is also problematic. It is conmutative, associative, there is a neutral element (1), and it is distributive. But there is not inverse either (equation $nx = 1$, in general, has no solution in $\mathbb{N}$). This means that we cannot define a division in $\mathbb{N}$ (this is another way of saying that divisions in $\mathbb{N}$ have, in general, a nonzero remainder).

Natural numbers do satisfy all axioms of order, so $\mathbb{N}$ is a totally ordered set. Not just that: since every subset of $\mathbb{N}$ contains a first element, we say that the order is *perfect* (or that $\mathbb{N}$ is *well-ordered*).

1.2.2 Induction principle

The recursive construction of natural numbers leads to an important type of mathematical proof: proofs by *induction*. The idea is that we have a proposition, $p(n)$, which is supposed to be valid for all $n \in \mathbb{N}$, and we want to prove it. Then all we need to do is to prove these two alternative propositions:

1. Proposition $p(1)$ is true.
2. If proposition $p(n)$ is true then necessarily proposition $p(n+1)$ is also true.

■ **Example 1.4** We will prove by induction this famous formula (first obtained by Gauss):

$$1+2+\cdots+n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}. \tag{1.1}$$

This is a proposition that is supposed to hold for all natural numbers.

So let us have a look at $p(1)$:

$$1 = \frac{1(1+1)}{2},$$

which is evindently true. This proves the first induction step. As for the second, let us assume that (1.1) holds for a given $n \in \mathbb{N}$ and let us add $n+1$ to both sides of the equation:

$$1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}.$$

This is equation (1.1) with n replaced by $n+1$. Thus we have *derived* proposition $p(n+1)$ out of proposition $p(n)$, and therefore, if $p(n)$ is true so must be $p(n+1)$.

The formula is proven for all $n \in \mathbb{N}$. ■

R It is important to realise the difference between the proposition

$p(n)$ is true for all $n \in \mathbb{N}$,

and the proposition

$p(n)$ is true.

The former is what we want to prove. The latter is valid for *that* particular n and no other, and is what we are *assuming* as the second induction step in order to actually prove $p(n+1)$.

Exercise 1.1 Prove by induction the formulas

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}.$$

■

1.2.3 Integer numbers

The set of integer numbers is introduced to satisfy all the addition axioms of the definition of field. Without being rigorous, the set of integers numbers is $\mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$. In other words, we complete the set $\mathbb{N}$ with 0 and with another copy of $\mathbb{N}$, which we label with a minus sign $(-\mathbb{N})$ and call *negative numbers*. Thus, $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$.

Addition can be defined casewise:

1. Natural numbers follow the standard addition rule.
2. To add two negative numbers we add the numbers without sign and put a minus sign to the result.
3. To add $x \in \mathbb{N}$ and $y \in -\mathbb{N}$ we ignore signs, substract the smallest from the largest, and put the original sign of the largest to the result (with the caveat that 0 has no sign).

If you find these rules bizarre and lacking common sense, think of an alternative, more practical definition. Assume that $\mathbb{Z}$ is an infinite storey building, 0 being the ground floor, 1 the first floor, 2 the second floor, etc, and likewise -1 is the first basement (underneath the ground floor), -2 the second basement (underneath the first basement), etc. Now think of addition as the result of moving through floors in a lift. The first number of the addition will be the floor we start off from; the second number is the number of floors that the lift will move up (if sign is positive) or down (if sign is negative). Thus $3+(-2) = 1$ because going down two floors from the 3rd floor we end up in the 1st one. Similarly, $2+(-3) = -1$ because going down three floors from the 2nd one we end up in the 1st basement. And $(-1)+(-2) = -3$ means that starting from the 1st basement and going down two floors we end up in the 3rd basement.

Multiplication of integers follows the rule of multiplication of natural numbers, ignoring signs, and then the sign of the result follows the sign rules

×/÷	+	−
+	+	−
−	−	+

Again it is not difficult to understand why we have to adopt these rules. We do not need to justify the $+\cdot+=+$ rule, because that is the standard multiplication in $\mathbb{N}$. As for $-\cdot+=-$ (or $+\cdot-=-$), consider the product $(-1)\cdot 3$. The standard way to interpret this multiplication is 'add -1 three times', and if we do that we obtain

$$(-1)\cdot 3 = (-1)+(-1)+(-1) = -3.$$

Now think of this sequence of operations:

$$\begin{aligned}
&(-1)\cdot 3 = -3,\\
&(-1)\cdot 2 = -2,\\
&(-1)\cdot 1 = -1,\\
&(-1)\cdot 0 = 0,\\
&(-1)\cdot(-1) = ?
\end{aligned}$$

What number should we put in place of the question mark? Well, if we continue the sequence we observe on the right-hand sides of these equations, logically it would be 1. Thus $(-1)\cdot(-1) = 1$, and this justifies the rule $-\cdot-=+$.

It is not difficult to prove that all addition axioms are satified by $\mathbb{Z}$, so substraction is well defined. However, we have not added any new axiom to the multiplication, for $zx = 1$ is still an equation without solution for most $z \in \mathbb{Z}$.

$\mathbb{Z}$ is also a totally ordered set, although it has lost the perfect order of $\mathbb{N}$ because it lacks a first element.

1.2.4 Rational numbers

Rational numbers are the response to the search for a solution of the equation $ax = 1$ for any number a, and the first ordered field. They are defined as $\mathbb{Q} = \{n/m : n, m \in \mathbb{Z}, m \neq 0\} = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. The fraction n/m is the solution of the equation $mx = n$.

There are infinitely many fractions that can solve the equation $mx = n$ ($m \neq 0$), namely n/m, $2n/2m$, $(-n)/(-m)$, etc., or if $n = pk$ and $m = pl$, also k/l. Thus all these fractions are "equivalent" in the sense that they represent the same rational number. So rational numbers are actually equivalence classes of fractions, where two fractions n_1/m_1 and n_2/m_2 are said to be equivalent if $n_1 m_2 = n_2 m_1$.

Rational numbers can also be ordered. If $r_1, r_2 \in \mathbb{Q}$ and we take representative fractions of them $r_1 = n_1/m_1$ and $r_2 = n_1/m_2$ (where $m_1 > 0$ and $m_2 > 0$), then $r_1 \leqslant r_2$ if $n_1 m_2 \leqslant n_2 m_1$, and $r_1 > r_2$ otherwise. As it can be seen, the order is linear.

$\mathbb{Q}$ is dense in itself, meaning that there is a rational number between any two distinct rational numbers. Clearly, if $r_1 < r_2$ are two rational numbers, $r = (r_1 + r_2)/2$, also a rational number, is between them two.

We might impose a well-ordering on $\mathbb{Q}$ if we wanted to, by the following procedure: start with 0, 1, -1; then append all *irreducible* fractions p/q such that $-2 \leqslant p, q, \leqslant 2$; then append those such that $-3 \leqslant p, q, \leqslant 3$; and so on and so forth. The result is that $\mathbb{Q}$ can be explicitly displayed as

$$\mathbb{Q} = \left\{0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{3}{2}, -\frac{3}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, 4, -4, \frac{4}{3}, -\frac{4}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{3}{4}, \dots\right\}.$$

A very important consequence of this fact is that there are as many numbers in $\mathbb{N}$ as in $\mathbb{Q}$ (because ordering $\mathbb{Q}$ as above means that a one-to-one correspondence can be established between both sets).

1.2.5 Real numbers

Despite the density of rational numbers, there are "holes" in between. For instance, the length of the diagonal of a square of unit side is $\sqrt{2}$, not a rational number. One can find rational numbers larger or smaller, and arbitrarily close to $\sqrt{2}$, but never a rational number that is exactly $\sqrt{2}$.

R The proof of this fact is an elegant example of *reductio ad absurdum*. Suppose $\sqrt{2}$ is rational and let n/m be its irreducible fraction (n and m have no common factors that can be simplified). Squaring the expression we obtain $2m^2 = n^2$, so n^2 is even, and therefore so is n (because if n were odd, n^2, being the product of two odd numbers, would be odd as well). Thus $n = 2k$. Substituting in this expression $2m^2 = 4k^2$, which we can simplify to $m^2 = 2k^2$. This implies that m^2 is even and therefore so is m. But that is impossible because m and n should not have any common factor.

So, from the assumption that $\sqrt{2}$ is rational we arrive at a contradiction, therefore the assumption is false and $\sqrt{2}$ is an *irrational* number.

Further insight on irrational numbers can be gained introducing the decimal representation of rational numbers. We can represent rationals as decimal expressions which contain an integer number, a decimal point, and a infinite sequence of digits. For example

$$\frac{1}{2} = 0.5\underline{0}0000\dots, \quad \frac{1}{3} = 0.\underline{3}33333\dots, \quad \frac{7}{6} = 1.1\underline{6}6666\dots, \quad \frac{23}{13} = 1.\underline{769230}769230\dots$$

What all these expressions have in common is that the infinite sequence of digits on the right of the decimal point is eventually periodic. The period may be longer or shorter, but there is always a definite period. Irrational numbers are all sequences of *aperiodic* decimal expressions. For instance,

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480 7317\ldots$$

We could continue calculating decimals of this number but the sequence would never become periodic.

We can construct the following bracketing of any irrational number by sequences of pairs of rational numbers:

$$\begin{array}{ccccc} 1.41 & < & \sqrt{2} & < & 1.42 \\ 1.414 & < & \sqrt{2} & < & 1.415 \\ 1.4142 & < & \sqrt{2} & < & 1.4143 \\ 1.41421 & < & \sqrt{2} & < & 1.41422 \\ \vdots & & \vdots & & \vdots \end{array}$$

This procedure allows us to have a mental representation of any irrational number as that number that would be bracketed by the two infinite sequences. This is how rational numbers can be "completed" by filling in the holes they leave. This is how *real numbers* are constructed.

Real numbers inherit all properties from their rational "approximants" and thus form an ordered field. But on top of that, the set they form, $\mathbb{R}$, has a significant difference when compared to $\mathbb{Q}$: it is complete. We will later come back to this important property of real numbers, but we want to emphasise here that the main implication of this property is that there is a one-to-one correspondence between real numbers and points in a straight line: there is a real number (and only one) to represent every scalar magnitude of any type. This is what makes real numbers so useful. It is also the reason why we often refer to $\mathbb{R}$ as *the real line.*

R A remarkable observation is that $\mathbb{R}$ is a set much bigger than $\mathbb{Q}$ —which, as we kown, is as "large" as $\mathbb{N}$ and no more. As a matter of fact, we will show that there are more real numbers between 0 and 1 than there are in $\mathbb{N}$. To prove it we also proceed through *reductio ad absurdum.* Let us us assume that we can order all those numbers in a sequence (as we did for $\mathbb{Q}$, for instance). To avoid being too abstract, just imagine that this is the beginning of the list:

$$\begin{aligned} r_1 &= 0.\underline{0}1004872657892653490023\ldots \\ r_2 &= 0.9\underline{8}296480010826402228946\ldots \\ r_3 &= 0.61\underline{1}55551000102988922200\ldots \\ r_4 &= 0.111\underline{1}1989887811110101010\ldots \\ &\vdots \end{aligned}$$

We have underlined the first decimal digit of the first number, the second decimal of the second numbers, and so on. We will construct a real number between 0 and 1 as follows: take for the first decimal a number different from the number underlined in r_1; for the second decimal a number different from the number underlined in r_2; and so on. For instance, $r = 0.2790\ldots$. This number is not in the list, because it is different —by construction— to every one of the elements of the list. However, it is a real number between 0 and 1 and therefore should be in the list!

The conclusion from this contradiction is that our assumption is false: real numbers cannot be listed. In other words, there is not a one-to-one correspondence between $\mathbb{R}$ and $\mathbb{N}$ because there are more real numbers than natural numbers. Real numbers *cannot be counted.* Clearly this means that irrational numbers are the ones that cannot be counted, so there are many more irrational numbers that rational numbers!

1.3 Absolute value, bounds, and intervals

In order to quantify how big or small are real numbers we have to get rid of their sign. This leads to introduce their *absolute value* or *magnitude,* which we denote $|x|$. In particular, $|x-y|$ quantifies the difference between x and y, and therefore their *distance* in the real line.

Definition 1.3.1 — Absolute value. For any $x \in \mathbb{R}$ we define its *absolute value* as

$$|x| = \max\{x, -x\} = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

The absolute value has some important properties:

Proposition 1.3.1 For all $x, y \in \mathbb{R}$,
1. $|x| \geqslant 0$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|xy| = |x||y|$.
4. $|x+y| \leqslant |x| + |y|$.
5. $\big||x| - |y|\big| \leqslant |x-y|$.

All of them are easy to check except 5. But this one follows from 4. To prove it we observe that

$$|x| = |x-y+y| \leqslant |x-y| + |y| \quad \Rightarrow \quad |x| - |y| \leqslant |x-y|.$$

But on the other hand,

$$|y| = |y-x+x| \leqslant |y-x| + |x| \quad \Rightarrow \quad |y| - |x| \leqslant |y-x| = |x-y|.$$

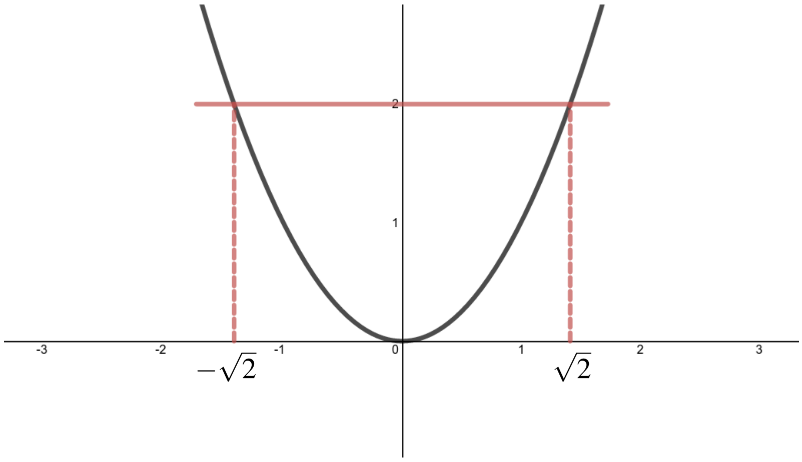
If both numbers $|x| - |y|$ and $|y| - |x| = -(|x| - |y|)$ are not larger than $|x-y|$, then $\big||x| - |y|\big| \leqslant |x-y|$.

Definition 1.3.2 — Bounds. Let A be a nonempty subset of $\mathbb{R}$, and let $x \in \mathbb{R}$.
1. x is an **upper bound** of A if $x \geqslant a$ for all $a \in A$ (A is then **bounded from above**).
2. x is an **lower bound** of A if $x \leqslant a$ for all $a \in A$ (A is then **bounded from below**).
3. A is **bounded** if it is bounded from above and from below.
4. x is the **supremum** of A ($x = \sup A$) if it is its *least upper bound* ($\sup A = \infty$ if A is not bounded from above).
5. x is the **infimum** of A ($x = \inf A$) if it is its *greatest lower bound* ($\inf A = -\infty$ if A is not bounded from below).
6. If $\sup A \in A$ it is called **maximum** ($\max A$).
7. If $\inf A \in A$ it is called **minimum** ($\min A$).

We have not properly defined real numbers. In particular, we have not provided a rigorous statement of the property of *completeness* that characterises $\mathbb{R}$ ($\mathbb{R}$ is the only complete ordered field). There are many ways of introducing this axiom, and one in particular relies on the idea of bracketing intervals intuitively introduced in the previous section. There is an important result that turns out to be equivalent to the completeness axiom, and which calculus uses profusely.

Theorem 1.3.2 — Completeness theorem. Let $A \subset \mathbb{R}$ be nonempty ($A \neq \varnothing$). The following properties hold:
- **(i) Supremum property:** If A is bounded from above then it has a supremum.
- **(ii) Infimum property:** If A is bounded from below then it has an infimum.

■ **Example 1.5** Consider the set $A = \{x \in \mathbb{R} : x^2 < 2\}$. This set is the portion of the real line for which the parabola $y = x^2$ lies strictly below the horizontal line $y = 2$, as illustrated by this picture:



From the figure it is clear that $A = \{x \in \mathbb{R} : -\sqrt{2} < x < \sqrt{2}\}$, because $x = \pm\sqrt{2}$ are the two points where $y = x^2$ meets $y = 2$. So $\sqrt{2}$ is an upper bound, and there can be no smaller bound, so $\sup A = \sqrt{2}$. Likewise, $\inf A = -\sqrt{2}$. (They are not maximum and minimum because they do not belong to A.)

Notice that if we consider instead the subset of $\mathbb{Q}$ defined as $A' = \{x \in \mathbb{Q} : x^2 < 2\}$, even though the definition looks similar to that of A, this set A' has neither a supremum nor an infimum in $\mathbb{Q}$. Any upper bound $r \in \mathbb{Q}$ of A' must necessarily be $r > \sqrt{2}$, and we can always find another $r' \in \mathbb{Q}$ such that $\sqrt{2} < r' < r$. Therefore no least upper bound can exist. ■

Definition 1.3.3 — Intervals. The following subsets of $\mathbb{R}$ are globally referred to as intervals:
Open interval: $(a,b) = \{x \in \mathbb{R} : a < x < b\}$.
Closed interval: $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$.
Semiopen intervals:

$$(a,b] = \{x \in \mathbb{R} : a < x \leqslant b\},$$
$$[a,b) = \{x \in \mathbb{R} : a \leqslant x < b\}.$$

Infinite intervals:

$$(-\infty,b] = \{x \in \mathbb{R} : x \leqslant b\},$$
$$(-\infty,b) = \{x \in \mathbb{R} : x < b\},$$
$$[a,\infty) = \{x \in \mathbb{R} : a \leqslant x\},$$
$$(a,\infty) = \{x \in \mathbb{R} : a < x\},$$
$$(-\infty,\infty) = \mathbb{R}.$$

■ **Example 1.6** Consider the set $A = \{x \in \mathbb{R} : |x| \leqslant 3\}$. Let us discuss which real numbers belong to A. To do that we need to distinguish the cases (a) $x \geqslant 0$ and (b) $x < 0$.

(a) If $x \geqslant 0$ then condition $|x| \leqslant 3$ reads $x \leqslant 3$. Thus the interval $[0,3]$ is part of A.
(b) If $x < 0$ the condition $|x| \leqslant 3$ reads $-x \leqslant 3$, or equivalently, $x \geqslant -3$. Therefore $[-3,0)$ is also part of A.

Summarising, $A = [-3,0) \cup [0,3] = [-3,3]$. ■

■ **Example 1.7** Consider the set $B = \{x \in \mathbb{R} : (x-1)(x-2)(x-3) < 0\}$. The condition that defines B depends on the sign of the product three factors, so we need to know the sign of each of them. These signs depend on whether (a) $x < 1$, (b) $1 < x < 2$, (c) $2 < x < 3$, or (d) $3 < x$. In cases (a) and (c) there is an odd number of negative factors, so the product is negative, whereas in cases (b) and (d) there is an even number of them, so the product is positive. Thus $B = (-\infty,1) \cup (2,3)$. ■

Problems

Problem 1.1 Given the real numbers $0 < a < b$ and $c > 0$, prove the inequalities

$$\text{(a)}\ \ a < \sqrt{ab} < \frac{a+b}{2} < b, \qquad \text{(b)}\ \ \frac{a}{b} < \frac{a+c}{b+c}.$$

Problem 1.2 Prove that $|a+b| = |a| + |b|$ if and only if $ab \geqslant 0$.

Problem 1.3 Prove that

$$\text{(a)}\ \ \max\{x,y\} = \frac{x+y+|x-y|}{2}, \qquad \text{(b)}\ \ \min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

Problem 1.4 Find, using the absolute value, a formula to express the function

$$\varphi(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Problem 1.5 Factor out the following expressions of $n \in \mathbb{N}$, so that the corresponding statements become self-evident:

(a) $n^2 - n$ is even,
(b) $n^3 - n$ is a multiple of 6,
(c) $n^2 - 1$ is a multiple of 8 when n is odd.

Problem 1.6 Prove by induction the following statements valid for all $n \in \mathbb{N}$:

(a) $a^n - b^n = (a-b)\sum_{k=1}^{n} a^{n-k}b^{k-1}$ for all $n \in \mathbb{N}$,

(b) $n^5 - n$ is a multiple of 5 for all $n \in \mathbb{N}$,

(c) $(1+x)^n \geqslant 1 + nx$ if $x \geqslant -1$.

Problem 1.7 Prove by induction the following statements valid for all natural numbers $n > 1$:

(a) $n! < \left(\dfrac{n+1}{2}\right)^n$,

(b) $2!4!\cdots(2n)! > \big[(n+1)!\big]^n$,

(c) $1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$.

HINT: In (a) use the inequality $\left(1 + \dfrac{1}{n+1}\right)^{n+1} > 2$, valid for all $n \in \mathbb{N}$. In (b) prove first that $(2n+2)! > (n+2)^n(n+2)!$.

Problem 1.8

(a) Show, with an example, that the sum of two irrational numbers can be rational.
(b) Show, with an example, that the product of two irrational numbers can be rational.
(c) Is it possible to find irrational numbers x and y such that $x^y \in \mathbb{Q}$?

Problem 1.9 Prove that

(a) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$,

(b) $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square (HINT: write $n = k^2 r$, where r does not contain any square factor),

(c) $\sqrt{n-1} + \sqrt{n+1} \notin \mathbb{Q}$ for all $n \in \mathbb{N}$.

Problem 1.10 Prove the identity, valid for all $x \in \mathbb{R}$,

$$\left(\frac{x+|x|}{2}\right)^2 + \left(\frac{x-|x|}{2}\right)^2 = x^2.$$

Problem 1.11 Identify the following sets:

(i) $A = \{x \in \mathbb{R} : |x-3| \leqslant 8\}$,
(ii) $B = \{x \in \mathbb{R} : 0 < |x-2| < 1/2\}$,
(iii) $C = \{x \in \mathbb{R} : x^2 - 5x + 6 \geqslant 0\}$,
(iv) $D = \{x \in \mathbb{R} : x^3(x+3)(x-5) < 0\}$,
(v) $E = \left\{x \in \mathbb{R} : \dfrac{2x+8}{x^2+8x+7} > 0\right\}$,
(vi) $F = \left\{x \in \mathbb{R} : \dfrac{4}{x} < x\right\}$,
(vii) $G = \{x \in \mathbb{R} : 4x < 2x+1 \leqslant 3x+2\}$,
(viii) $H = \{x \in \mathbb{R} : |x^2 - 2x| < 1\}$,
(ix) $I = \{x \in \mathbb{R} : |x-1||x+2| = 10\}$,
(x) $J = \{x \in \mathbb{R} : |x-1| + |x+2| > 1\}$.

Problem 1.12 Given real numbers $a < b$ we define, for each $t \in \mathbb{R}$, the real number $x(t) = (1-t)a + tb$. Identify the following sets:

(i) $A = \{x(t) : t = 0, 1, 1/2\}$,
(ii) $B = \{x(t) : t \in (0,1)\}$,
(iii) $C = \{x(t) : t < 0\}$,
(iv) $D = \{x(t) : t > 1\}$.

Problem 1.13 Find supremum and infimum (deciding whether they are maximum and minimum respectively) of the following sets:

(i) $A = \{-1\} \cup [2,3)$,
(ii) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0,1]$,
(iii) $C = \{2 + 1/n : n \in \mathbb{N}\}$,
(iv) $D = \{(n^2+1)/n : n \in \mathbb{N}\}$,
(v) $E = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$,
(vi) $F = \{x \in \mathbb{R} : (x-a)(x-b)(x-c)(x-d) < 0\}$, with $a < b < c < d$ given real numbers,
(vii) $G = \{2^{-p} + 5^{-q} : p, q \in \mathbb{N}\}$,
(viii) $H = \{(-1)^n + 1/m : n, m \in \mathbb{N}\}$,