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Calculus I

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Unit 10. Fundamental Theorem of Calculus

10. Fundamental Theorem of Calculus

Integration is a device that was invented to calculate areas of figures limited by curved sides. The idea can be traced back at least to Archimedes. He is well known —among many other things— by calculating the area of a circle of unit diameter, A, in terms of its perimeter, π (π is the initial of περίμετρος = *perimeter*), obtaining the celebrated formula $A = \pi/4$. He did that by using two sequence of polygons, both circumscribed to and inscribed in the circumference, and then taking the limit of the number of sides going to infinity (see Figure 10.1).

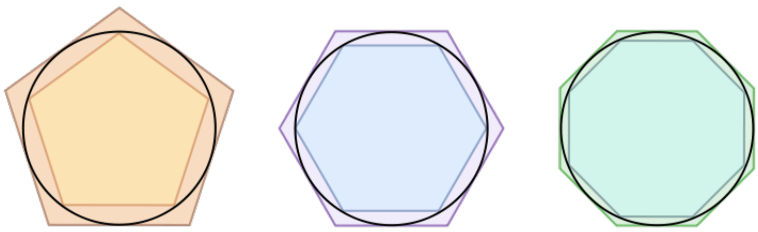
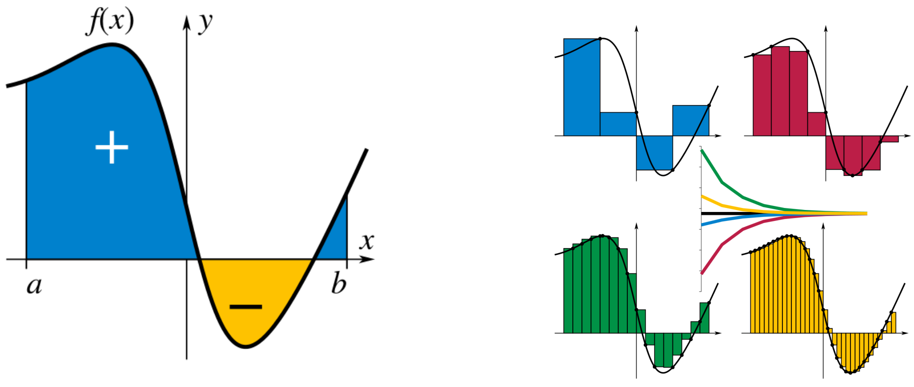


Figure 10.1: Archimedes's construction to obtain the relation between the area and the perimeter of a circle.

A similar idea was employed to obtain the area under more complicated curves. If we define a signed area as in Figure 10.2(a) (i.e., it adds if $f(x) > 0$ and substracts if $f(x) < 0$), the problem is how to calculate the total area enclosed by a curved within a given interval. Following Archimedes, one way to estimate that area is to approximate it as a sum of rectangles, as in Figure 10.2(b). In the limit when the width of these rectangles goes to zero we obtain the value of the seeked area.

■ **Example 10.1** As an example of this procedure, let us calculate, using this method, the area below the curve $f(x) = x^2$ within the interval $[0, a]$. To do that, we divide the interval in n rectangles of width a/n and heights $(ak/n)^2$, with $k = 1, 2, \dots, n$. The areas of these rectangles will then be



(a) (b)

Figure 10.2: (a) Area “under” a curve: above the X axis area has a positive sign and below the X axis has a negative sign. (b) Approximations to that area as sums of thiner and thiner rectangles.

a^3k^2/n^3. This yields the following approximation to the area:

$$A_n = \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{a^3}{n^3} \sum_{k=1}^{n} k^2.$$

It is a know result that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

thus

$$A_n = a^3 \frac{n(n+1)(2n+1)}{6n^3}.$$

Therefore

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} a^3 \frac{n(n+1)(2n+1)}{6n^3} = \frac{a^3}{3}$$

is the area we are seeking. ■

10.1 Riemann’s integral

The problem with the heuristic idea exposed above is that, for that procedure to make sense, the result should not depend on the division in rectangles that we propose. In other words, irrespective of whether we choose all rectangles to have the same or different widths, the limit process should yield the same area. Thus we need a more rigorous construction and limit process.

To this purpose, given an interval $[a,b]$ we will define a **partition** of the interval as as the set $P = \{x_0, x_1, \dots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Now, for any function f bounded in $[a,b]$, if we define

$$m_i \equiv \inf_{x_{i-1}\leqslant x\leqslant x_i} f(x), \qquad M_i \equiv \sup_{x_{i-1}\leqslant x\leqslant x_i} f(x), \tag{10.1}$$

then the (signed) area between the X axis and $f(x)$ within the interval $[x_{i-1}, x_i]$ —provided it can be defined— will be bounded from below by $m_i(x_i - x_{i-1})$ and from above by $M_i(x_i - x_{i-1})$ —the areas of two rectangles (see Figure 10.3). Thus, the two numbers

$$L(f,P) = \sum_{i=1} m_i(x_i - x_{i-1}), \qquad U(f,P) = \sum_{i=1} M_i(x_i - x_{i-1}), \tag{10.2}$$

respectively called **lower sum** and **upper sum** of f with respect to the partition P, will be an upper and a lower bound to the (signed) area between $f(x)$ and the X axis within the interval $[a,b]$. By construction $L(f,P) \leqslant U(f,P)$.

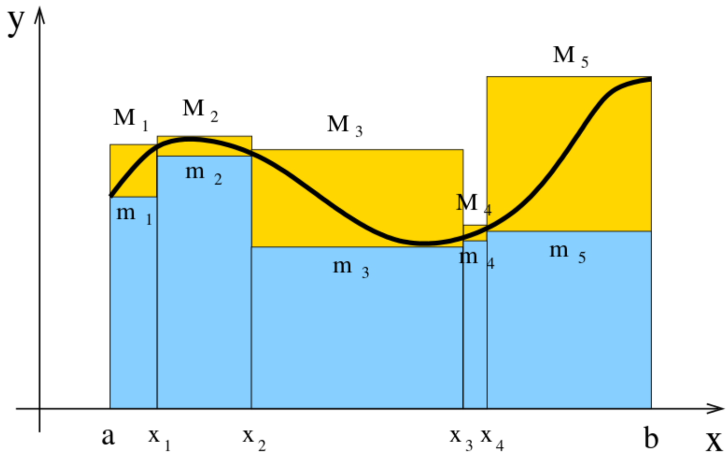


Figure 10.3: Definition of the upper sum and lower sum for a function $f(x)$ with respecto to a partition of the interval $[a,b]$. The (signed) area between the X axis and $f(x)$ is bounded between them two.

Partitions can be defined by adding more points to it. Thus, Q is a **refinement** of P if $P \subset Q$. Upon refining partitions we increase the lower sum and decrease the upper sum, i.e.,

$$L(f,P) \leqslant L(f,Q), \qquad U(f,Q) \leqslant U(f,P).$$

Accordingly, if P_1 and P_2 are two partitions of $[a,b]$, then $Q = P_1 \cup P_2$ will be a refinement of both of them and therefore

$$L(f,P_1) \leqslant L(f,Q) \leqslant U(f,Q) \leqslant U(f,P_2).$$

In other words, $L(f,P_1) \leqslant U(f,P_2)$ irrespective of the partitions P_1 and P_2.

This is summarised in the statement

$$\sup_P L(f,P) \leqslant \inf_P U(f,P). \tag{10.3}$$

This led Riemann to invent the following definition:

Definition 10.1.1 — Integral. A function f bounded in $[a,b]$ is **integrable** in $[a,b]$ if

$$\sup_P L(f,P) = \inf_P U(f,P) = \int_a^b f. \tag{10.4}$$

The number $\int_a^b f$ is known as the (Riemann's) **integral** of f in $[a,b]$.

R It is customary to use Leibniz's notation for the integral and write

$$\int_a^b f = \int_a^b f(x)\,dx.$$

This notation reminds the definition of the integral as a sum (hence the sign $\int$) of the areas of rectagles of with dx and height $f(x)$, for all $a \leqslant x \leqslant b$.

■ **Example 10.2** Not all bounded functions can be integrated. For instance the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

does not satisfy the definition because for every partition P of the interval $[a,b]$ we have $L(f,P) = 0$ and $U(f,P) = b - a$ —because every subterval $[x_{k-1}, x_k]$ contains both rational and irrational numbers. ■

■ **Example 10.3** The function

$$f(x) = \begin{cases} 1, & x = \frac{1}{2}, \\ 0, & x \neq \frac{1}{2}, \end{cases}$$

can be integrated in e.g. $[0,1]$. Let P be any partition of that interval. Then $L(f,P) = 0$ because in every interval of P f takes the value 0. On the other hand, $U(f,P) = \Delta x$, where Δx is the length of the interval containing the point $x = 1/2$. Since by refining the partition Δx can be made arbitrarily small,

$$L(f,P) = \inf_P U(f,P) = 0 \qquad \Rightarrow \qquad \int_a^b f = 0.$$

■

An important result that justifies the heuristic construction is this:

Theorem 10.1.1 The bounded function f is integrable in $[a,b]$ if and only if there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n).$$

In other words, to prove the existence of an integral we simply have to take a partition P_n of the interval $[a,b]$ into n equal segments, compute $L(f,P_n)$ and $U(f,P_n)$ and take the limits.

Exercise 10.1 Transform Example 10.1 into a rigorous proof that $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}$. ■

The full characterisation of the set of functions that can be integrated according to Riemann's definition is out of the scope of this course. However, this set includes important classes of functions worth mentioning:

Theorem 10.1.2 If f is continuous in $[a,b]$ then it is integrable in $[a,b]$.

The idea of the proof of this result is that continuous functions have the property that the difference between their maximum and minimum values in a closed interval is smaller the smaller the interval. This means that we can make the difference between $L(f,P)$ and $U(f,P)$ arbitrarily small by simply refining the partition sufficiently.

Theorem 10.1.3 If f is monotonic in $[a,b]$ then it is integrable in $[a,b]$.

Proof. Let us assume that f is increasing (the proof is analogous for decreasing functions). The idea of the proof is that, within the interval $[x_{i-1}, x_i]$, the maximum of f is $f(x_i)$ and the minimum is $f(x_{i-1})$. Thus, if P_n is the partition of $[a,b]$ into n equal size intervals,

$$L(f,P_n) = \sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n}, \qquad U(f,P_n) = \sum_{i=1}^{n} f(x_i)\frac{b-a}{n},$$

and therefore

$$0 \leqslant U(f,P_n) - L(f,P_n) = \frac{b-a}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] = \frac{b-a}{n}[f(b) - f(a)] \xrightarrow[n\to\infty]{} 0.$$

■

Notice that monotonic functions need not be continuous, so this result is not contained in the previous one.

10.2 Properties of the integral

Theorem 10.2.1 Let f and g be two integrable functions in $[a,b]$. Then the following properties hold:

(i) $\displaystyle\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ for all $\alpha, \beta \in \mathbb{R}$ **linearity**

(ii) $\displaystyle\int_a^b f \leqslant \int_a^b g$ whenever $f \leqslant g$ in $[a,b]$ **boundedness**

(iii) $|f|$ is integrable in $[a,b]$ and $\displaystyle\left|\int_a^b f\right| \leqslant \int_a^b |f|$ **absolute integrability**

A consequence of (ii) is that if $f \geqslant 0$ then $\displaystyle\int_a^b f \geqslant 0$.

Another consequence is that if $M = \sup\limits_{x\in[a,b]} f(x)$ and $m = \inf\limits_{x\in[a,b]} f(x)$, then

$$m(b-a) \leqslant \int_a^b f \leqslant M(b-a). \tag{10.5}$$

Theorem 10.2.2 — Interval additivity. Given $a < b < c$, function f is integrable in $[a,c]$ if and only if it is integrable in $[a,b]$ and $[b,c]$. Besides

$$\int_a^c f = \int_a^b f + \int_b^c f. \tag{10.6}$$

Notice that this formula implies

$$\int_a^b f = \int_a^c f - \int_b^c f,$$

so interval additivity will be preserved beyond the constraint $a < b < c$ if we *define*

$$\int_c^b f = -\int_b^c f. \tag{10.7}$$

10.3 Riemann's sums

Let f be a bounded function in $[a,b]$. For any partition P of this interval the expression

$$S(f,P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}), \tag{10.8}$$

for any choice of points $x_{i-1} \leqslant c_i \leqslant x_i$ is referred to as a **Riemann's sum.**

It is clear from the definition that Riemann's sums satisfy $L(f,P) \leqslant S(f,P) \leqslant U(f,P)$. Therefore, if f is integrable in $[a,b]$ and $\{P_n\}_{n=1}^{\infty}$ is a sequence of partitions such that

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0,$$

then

$$\lim_{n\to\infty} S(f,P_n)=\int_a^b f. \tag{10.9}$$

This result is very useful in calculating some limits, as the examples illustrate.

■ **Example 10.4** Suppose we need to calculate the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}.$$

This limit does not define a series, because the terms in the sum change not only with k but also with n.

In order to calculate this limit we need to rewrite the sum as

$$\sum_{k=1}^{n}\frac{1}{n+k}=\sum_{k=1}^{n}\frac{1}{1+(k/n)}\cdot\frac{1}{n}.$$

The right-hand side is the expression of $S(f,P_n)$, where $f(x)=1/(1+x)$, $c_k=k/n$ and P_n is a partition of $[0,1]$ in n equal-size intervals. Since f is continuous —hence integrable—, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty} S(f,P_n)=\int_0^1\frac{dx}{1+x}.$$

■

■ **Example 10.5** Let us calculate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{1/n}.$$

If we denote the limit ℓ, then

$$\log\ell=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\log\left(1+\frac{k}{n}\right)=\lim_{n\to\infty} S(f,P_n),$$

where P_n is a partition of $[0,1]$ in n equal-size intervals and $f(x)=\log(1+x)$. Thus,

$$\log\ell=\int_0^1\log(1+x)\,dx.$$

■

10.4 Fundamental theorem of calculus

The basic idea of the connection between integrals and derivatives —the essence of the fundamental theorem of calculus— is this. Let us denote $A(x)$ the (signed) area between the X axis and the function f within the interval $[a,x]$. Suppose that we increase the inverval by a very small amount

h. In practical terms, we are enlarging the area by adding almost a rectangle of width h and height $\approx f(x)$. In other words,

$$A(x+h) \approx A(x) + f(x)h \qquad \Rightarrow \qquad f(x) \approx \frac{A(x+h)-A(x)}{h}.$$

If we now take the limit $h \to 0$ we obtain the connection $A'(x) = f(x)$. This is the basic result that both Newton and Leibniz were aware of and which renders calculus such a powerful tool.

We are going to obtain this result in a more rigorous way by using our definition of (Riemann's) integral.

To begin with, let us first prove that integrals always define continuous functions:

Theorem 10.4.1 If f is integrable in $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ defines a continuous function in $[a,b]$.

Proof. Take any point $c \in [a,b]$. Since f is integrable in $[a,b]$ it is also bounded, so let $M = \sup_{x\in[a,b]} |f(x)|$. Then

$$|F(x)-F(c)| = \left|\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right| = \left|\int_c^x f(t)\,dt\right| \leqslant \left|\int_c^x |f(t)|\,dt\right| \leqslant \left|\int_c^x M\,dt\right| = M|x-c|.$$

By the sandwich rule,

$$\lim_{x\to c} |x-c| = 0 \qquad \Rightarrow \qquad \lim_{x\to c} |F(x)-F(c)| = 0 \qquad \Rightarrow \qquad \lim_{x\to c} F(x) = F(c).$$

This proves that F is continuous at any $c \in [a,b]$. ■

Notice that this result requieres nothing from f apart from its integrability. In particular, f needs not be a continuous function.

■ **Example 10.6** Let

$$f(x) = \begin{cases} 0, & x \leqslant \frac{1}{2}, \\ 1, & x > \frac{1}{2} \end{cases}$$

be a function with a jump discointinuity at $x = 1/2$. Now, for any $x \leqslant 1/2$,

$$F(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0,$$

whereas for any $x > 1/2$,

$$F(x) = \int_0^x f(t)\,dt = \int_0^{1/2} f(t)\,dt + \int_{1/2}^x f(t)\,dt = \int_0^{1/2} 0\,dt + \int_{1/2}^x dt = x - \frac{1}{2}.$$

Thus,

$$F(x) = \begin{cases} 0, & x \leqslant \frac{1}{2}, \\ x - \frac{1}{2}, & x > \frac{1}{2}, \end{cases}$$

which is continuous everywhere. ■

Theorem 10.4.2 — First fundamental theorem of calculus. If f is continuous in $[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is differentiable in (a,b) and $F'(x) = f(x)$.

Proof. First of all

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Now, in the interval $[x,x+h]$ (or $[x+h,x]$ if $h<0$) f reaches its maximum M_h and minimum m_h values —as every continuous function in a closed interval. Then, if $h>0$

$$m_h h \leqslant \int_x^{x+h} f(t)\,dt \leqslant M_h h \qquad \Rightarrow \qquad m_h \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant M_h,$$

and if $h<0$

$$m_h(-h) \leqslant \int_{x+h}^{x} f(t)\,dt \leqslant M_h(-h) \qquad \Rightarrow \qquad m_h \leqslant \frac{1}{(-h)}\int_{x+h}^{x} f(t)\,dt \leqslant M_h$$

$$\Rightarrow \qquad m_h \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant M_h.$$

In any case, the number $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is an intermediate value between m_h and M_h. Any continuous function in a closed interval reaches all intermediate values between its maximum and its minimum, so there must be a point $c_h \in [x,x+h]$ (or in $[x+h,x]$ if $h<0$) such that

$$f(c_h) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Clearly $c_h \to x$ when $h \to 0$. Therefore

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c_h) = f(x).$$

■

The take-home message of this theorem is that integrals of functions are primitives of those functions. Here is the connection between differentiation and integration. From now on, calculating the area between the X axis and a given curve $f(x)$ is as simple as finding the right primitive of f. Actually, the problem is even easier: any primitive will do, because of this second version of the fundamental theorem of calculus:

Theorem 10.4.3 — Second fundamental theorem of calculus (Barrow's rule). If f is continuous in $[a,b]$ and G is any primitive of f in (a,b), then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Proof. According to the first version of this theorem $F(x) = \int_a^x f(t)\,dt$ is a primitive of f in (a,b). Therefore $G(x) = F(x) + c$. Now $F(a) = \int_a^a f(t)\,dt = 0$, hence $G(a) = F(a) + c = c$. In other words, $F(x) = G(x) - G(a)$. Then

$$\int_a^b f(x)\,dx = F(b) = G(b) - G(a).$$

■

R Often primitives are referred to as "indefinite integrals" and denoted $\int f(x)\,dx$, whereas integrals of the form $\int_a^b f(x)\,dx$ are called "definite integrals".

Corollary 10.4.4 If f is continuous in $[a,b]$ and g_1, g_2 are differentiable in (a,b) then

$$H(x) = \int_{g_1(x)}^{g_2(x)} f(t)\,dt \tag{10.10}$$

is also differentiable in (a,b) and

$$H'(x) = f\big(g_2(x)\big)g_2'(x) - f\big(g_1(x)\big)g_1'(x). \tag{10.11}$$

Proof. Let $F(x)$ be a primitive of $f(x)$ in (a,b). Then $H(x) = F\big(g_2(x)\big) - F\big(g_1(x)\big)$. Since F, g_1, g_2 are all differentiable, so is H. Finally, the derivative of H will be, by the chain rule,

$$H'(x) = F'\big(g_2(x)\big)g_2'(x) - F'\big(g_1(x)\big)g_1'(x) = f\big(g_2(x)\big)g_2'(x) - f\big(g_1(x)\big)g_1'(x)$$

because $F'(x) = f(x)$. ■

■ **Example 10.7** If

$$F(x) = \int_0^{x^3} \cos t\,dt,$$

then $F'(x) = 3x^2\cos(x^3)$. ■

Applying Barrow's rule we can obtain particular versions of the integration by parts and change of variable theorems:

Theorem 10.4.5 — Integration by parts. If f and g are two differentiable functions in (a,b), then

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx. \tag{10.12}$$

The symbol in the right-hand side is a short-hand for

$$f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a). \tag{10.13}$$

Theorem 10.4.6 — Change of variable. If g is continuous in $[a,b]$ and differentiable in (a,b), and f is continuous in $g\big([a,b]\big)$, then

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f\big(g(x)\big)g'(x)\,dx. \tag{10.14}$$

Proof. On the one hand, if F is a primitive of f then

$$\int_{g(a)}^{g(b)} f(u)\,du = F\big(g(b)\big) - F\big(g(a)\big).$$

On the other hand, by the chain rule,

$$\frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x),$$

therefore $F\big(g(x)\big)$ is a primitive of $f\big(g(x)\big)g'(x)$ and, according to Barrow's rule,

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = F\big(g(b)\big) - F\big(g(a)\big).$$

The result follows from the fact that the right-hand side is the same for both integrals. ■

■ **Example 10.8** Let us calculate the area of a circle of radius a. The equation of its circumference is $x^2+y^2=a^2$, from which we obtain $y=\pm\sqrt{a^2-x^2}$. Clearly the area between the X axis and the function $f(x)=\sqrt{a^2-x^2}$ within the interval $[-a,a]$ is half the area we want to calculate, therefore

$$A = 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx.$$

We can introduce the variable $t=x/a$, or $x=at$, so that $\dfrac{dx}{dt}=a$, and the limits $x=-a \;\to\; t=-1$ and $x=a \;\to\; t=1$. Thus

$$A = 2\int_{-1}^{1}\sqrt{a^2-a^2t^2}\,a\,dt = 2a^2\int_{-1}^{1}\sqrt{1-t^2}\,dt.$$

Let us now introduce a second change of variable: $t=\sin\theta$. Then $\dfrac{dt}{d\theta}=\cos\theta$, and the limits $t=-1 \;\to\; \theta=-\pi/2$ and $t=1 \;\to\; \theta=\pi/2$. The integral then becomes

$$A = 2a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = a^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = a^2\Bigg(\pi+\underbrace{\frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2}}_{=0}\Bigg) = \pi a^2.$$

■

■ **Example 10.9 — One last integration trick.** Suppose one has to compute the integral

$$I = \int_a^b f(x)\,dx.$$

A simple change of variable is given by $x=a+b-t$, which transforms the integral into

$$I = \int_a^b f(a+b-t)\,dt$$

(because $dx=-dt$ and $t=b$ for $x=a$ and $t=a$ for $x=b$). Then, an alternative way of writing the original integral is as an average of these two expressions, namely

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \big[f(x)+f(a+b-x)\big]\,dx.$$

As an illustrative example, let us calculate the integral

$$I = \int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}.$$

A first remark about this integral is that the integrand is a bounded, continuous function in $[0,\pi/2]$, because $\tan x \geqslant 0$ in this interval and, although it diverges when $x \to \left(\frac{\pi}{2}\right)^-$,

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1}{1+\sqrt{\tan x}} = 0.$$

A second remark is that performing this integral by any other standard method poses a real challenge (give it a try!) With this last trick though, it is a piece of cake.

According to the formula we have just derived,

$$I = \frac{1}{2}\int_0^{\pi/2} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{1}{1+\sqrt{\cot x}}\right) dx$$

because $\tan(\pi/2 - x) = \cot x$. But

$$\begin{aligned}\frac{1}{1+\sqrt{\tan x}} + \frac{1}{1+\sqrt{\cot x}} &= \frac{1+\sqrt{\cot x}+1+\sqrt{\tan x}}{(1+\sqrt{\tan x})(1+\sqrt{\cot x})} = \frac{2+\sqrt{\cot x}+\sqrt{\tan x}}{1+\sqrt{\tan x}+\sqrt{\cot x}+\sqrt{\tan x \cot x}} \\ &= \frac{2+\sqrt{\cot x}+\sqrt{\tan x}}{2+\sqrt{\tan x}+\sqrt{\cot x}} = 1,\end{aligned}$$

where we have just used the fact that $\tan x \cot x = 1$. Then

$$I = \frac{1}{2}\int_0^{\pi/2} dx = \frac{\pi}{4}.$$

■

Exercise 10.2 Use the method above to prove that

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}.$$

■

Problems

Problem 10.1 Find a continuous function f such that $f(0)=0$ and

$$f'(x)=\begin{cases}\dfrac{4-x^2}{(4+x^2)^2}, & x<0,\\ e^{\sqrt{x}}, & x>0.\end{cases}$$

Problem 10.2

(a) Prove that if f is odd then $\int_{-a}^{a} f(x)\,dx=0$.

(b) Prove that if f is even then $\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$.

(c) Calculate the integral $\int_6^{10}\sin\left(\sin\left((x-8)^3\right)\right)dx$.

Problem 10.3 Calculate the following limits:

(i) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$; (ii) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt[n]{e^{2k}}$; (iii) $\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n^2-k^2}}$.

Problem 10.4 Calculate $F(x)=\int_{-1}^{x} f(t)\,dt$, with $-1\leqslant x\leqslant 1$, for the following functions:

(i) $f(x)=|x|e^{-|x|}$;

(ii) $f(x)=|x-1/2|$;

(iii) $f(x)=\begin{cases}-1, & -1\leqslant x<0,\\ 1, & 0\leqslant x\leqslant 1;\end{cases}$

(iv) $f(x)=\begin{cases}x^2, & -1\leqslant x<0,\\ x^2-1, & 0\leqslant x\leqslant 1;\end{cases}$

(v) $f(x)=\begin{cases}1, & -1\leqslant x\leqslant 0,\\ x+1, & 0<x\leqslant 1;\end{cases}$

(vi) $f(x)=\begin{cases}1+x, & -1\leqslant x\leqslant -\frac12,\\ \frac12, & -\frac12<x<\frac12,\\ 1-x, & \frac12\leqslant x\leqslant 1;\end{cases}$

(vii) $f(x)=\max\left\{\sin(\pi x/2),\cos(\pi x/2)\right\}$.

Problem 10.5 Calculate the following integrals:

(i) $\int_0^{\log 2}\sqrt{e^x-1}\,dx$; (ii) $\int_1^2\frac{\sqrt{x^2-1}}{x}\,dx$.

Problem 10.6 Calculate the derivative of the following functions:

(i) $F(x)=\int_{x^2}^{x^3}\frac{e^t}{t}\,dt$;

(ii) $F(x)=\int_{-x^3}^{x^3}\frac{dt}{1+\sin^2 t}$;

(iii) $F(x)=\int_3^{\int_1^x \sin^3 t\,dt}\frac{dt}{1+\sin^6 t+t^2}$;

(iv) $F(x)=\int_2^{\exp\left\{\int_1^{x^2}\tan\sqrt{t}\,dt\right\}}\frac{ds}{\log s}$;

(v) $F(x)=\int_0^x x^2 f(t)\,dt$, with f continuous in $\mathbb{R}$;

(vi) $F(x)=\sin\left(\int_0^x\sin\left(\int_0^y\sin^3 t\,dt\right)dy\right)$.

Problem 10.7 Find the absolute maximum and minimum in the interval $[1,\infty)$ of the function

$$f(x)=\int_0^{x-1}\left(e^{-t^2}-e^{-2t}\right)dt.$$

HINT: $\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt=\sqrt{\pi}/2$.

Problem 10.8 Prove that the equation

$$\int_0^x e^{t^2}\,dt = 1$$

has a unique solution in $\mathbb{R}$ and that it can be found in the interval $(0,1)$.

Problem 10.9 Let $f(x)$ be a continuous function such that $f(x) > 0$ for all $0 \leqslant x \leqslant 1$, and consider the function

$$F(x) = 2\int_0^x f(t)\,dt - \int_x^1 f(t)\,dt.$$

Determine how many solutions the equation $F(x) = 0$ has in $[0,1]$.

Problem 10.10 Find and classify the local extrema within $(0,\infty)$ of the function

$$G(x) = \int_0^{x^2} \sin t e^{\sin t}\,dt.$$

Problem 10.11 Write the equation of the straight tangent to the curve

$$y = \int_{x^2}^{\sqrt{\pi}/2} \tan(t^2)\,dt$$

at the point $x = \sqrt[4]{\pi/4}$.

Problem 10.12 Given the function

$$f(x) = \begin{cases} \dfrac{e^x - 1 - x}{x^2}, & x < 0, \\[2ex] a + b\displaystyle\int_0^x e^{-t^4}\,dt, & x \geqslant 0, \end{cases}$$

calculate a and b so that it is continuous and differentiable.

Problem 10.13 Calculate the following limits:

(i) $\displaystyle\lim_{x\to 0} \frac{1}{x^3}\left(\int_0^x e^{t^2}\,dt - x\right)$;   (ii) $\displaystyle\lim_{x\to 0} \frac{\cos x}{x^4}\int_0^x \sin(t^3)\,dt.$

Problem 10.14 Calculate the two one-sided limits at $x = 0$ of the function

$$f(x) = \frac{1}{2x^3}\int_0^{x^2} \tan\sqrt{t}\,dt.$$

Problem 10.15 Consider the function $f(x) = \displaystyle\int_0^{x^2} \frac{\sin t}{t}\,dt.$

(a) Using the Taylor series of $\sin t$ in powers of t, find that of f in powers of x.

(b) Calculate $\displaystyle\lim_{x\to 0} \frac{f(x)}{1-\cos x}$.

(c) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} f(1/n)$.

Problem 10.16 Let $f(x) = \displaystyle\int_{-1/x}^{x} \frac{dt}{a^2+t^2}$. Determine, without computing the integral, for which values of a the function f is constant.

Problem 10.17 Consider the functions $f(x) = e^{x^2} - x^2 - 1$ and $g(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) Write the Taylor series of g in powers of x.
(b) Determine if g has a maximum, a minimum, or an inflection point at $x = 0$.

Problem 10.18
(a) Use the change of variable $t = \sin^2\theta$ to calculate the integral

$$\int_0^1 \arcsin\sqrt{t}\,dt.$$

(b) Consider the function

$$f(x) = \int_0^{\sin^2 x} \arcsin\sqrt{t}\,dt + \int_0^{\cos^2 x} \arccos\sqrt{t}\,dt.$$

Prove that $f(x) = c$, a constant, in the interval $[0, \pi/2]$.
(c) Determine the value of the constant c.

Problem 10.19 The equation

$$\int_0^{g(x)} \left(e^{t^2} + e^{-t^2}\right) dt = x^3 + 3\arctan x$$

defines an injective, differentiable function g in $\mathbb{R}$. Calculate:
(a) $g(0)$, $g'(0)$, and $\left(g^{-1}\right)'(0)$.
(b) $\lim\limits_{x\to 0} \dfrac{g^{-1}(x)}{g(x)}$.

Problem 10.20 Let $f : [-1,1] \mapsto \mathbb{R}$ be any integrable function.
(a) Prove that

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$$

HINT: Do the change of variables $y = \pi - x$.
(b) Calculate the integral

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$$

Problem 10.21 Let f be a differentiable function such that

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c.$$

Find $f(x)$ and the constant c.

Problem 10.22 Prove that

$$\int_0^x e^{t^2}\,dt \sim \frac{e^{x^2}}{2x} \quad (x \to \infty).$$

Problem 10.23 Let f be a function $n+1$ times differentiable in an interval I, and let $a, x \in I$. Assume that the integral defining the function

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt, \qquad n = 0, 1, \ldots$$

exists.
(a) Calculate $R_0(x)$.
(b) Integrating by parts, find a recurrence formula for $R_n(x)$.
(c) Solve the recurrence and interpret the result.