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# Calculus I

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## Unit 11. Geometric Applications of Integrals

# 11. Geometric Applications of Integrals

## 11.1 Area of flat figures

Given two functions $f$ and $g$ such that $f(x) \leqslant g(x)$ for all $x \in [a,b]$, we can obtain the area of the flat figure $S$ delimited by them and the vertical lines at $x = a$ and $x = b$ (see Figure 11.1(a)) as

$$\mathscr{A}(S) = \int_a^b \left[g(x) - f(x)\right] dx. \tag{11.1}$$

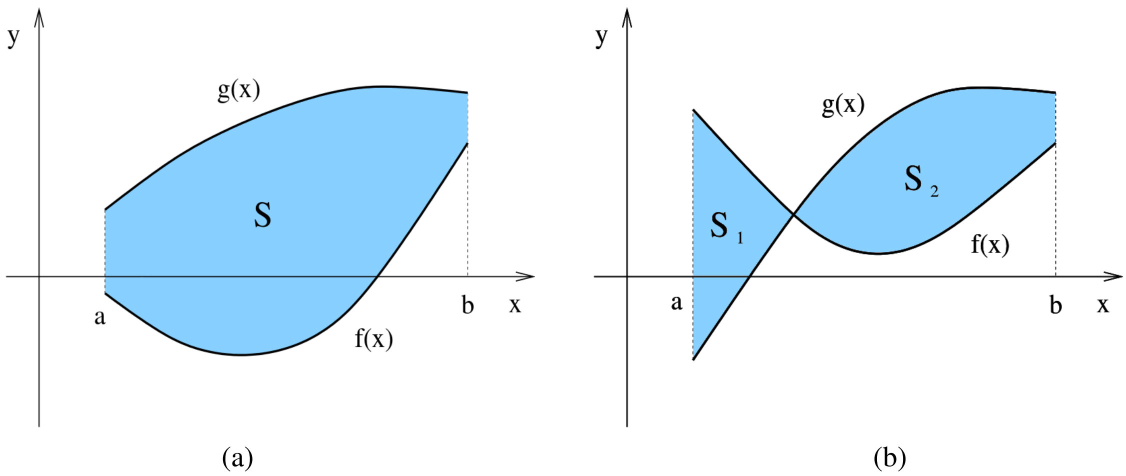



Figure 11.1: Flat figure delimited by the functions $f(x)$ and $g(x)$ and the vertical lines at $x = a$ and $x = b$. (a) Simple case where $f \leqslant g$. (b) Case in which $f$ and $g$ cross each other, so the figure is actually the union of several figures.

In a general case, where $f$ and $g$ can cross one or several times within the interval $[a,b]$, the figure $S$ is made of the union of several figures —joined at the crossing points (see Figure 11.1(b)). Strictly speaking we should then decompose the calculation between consecutive crossing points and apply formula (11.1) taking into account which function is the largest in each subinterval. This can be done automatically by extending formula (11.1) as

$$\mathscr{A}(S)=\int_a^b \left|g(x)-f(x)\right| dx. \tag{11.2}$$

■ **Example 11.1** Let us calculate the area between $f(x)=x(x-2)$ and $g(x)=x/2$ within the interval $[0,2]$. Since in that interval $f(x)\leqslant 0$ and $g(x)\geqslant 0$,

$$\begin{aligned}\mathscr{A}(S)&=\int_0^2\left(\frac{x}{2}-x(x-2)\right)dx=\int_0^2\left(\frac{x}{2}-x^2+2x\right)dx=\frac{x^2}{4}-\frac{x^3}{3}+x^2\Big|_0^2\\&=1-\frac{8}{3}+4=\frac{7}{3}.\end{aligned}$$

■

■ **Example 11.2** Let us now calculate the area between the curves $f(x)=x$ and $g(x)=x^3/4$ within the interval $[-1,2]$. First we need to find the crossing points:

$$x=\frac{x^3}{4} \qquad\Rightarrow\qquad x=0,\ x=\pm 2.$$

Between $-2$ and $0$ we have $g\geqslant f$, but between $0$ and $2$ the opposite holds. Thus,

$$\begin{aligned}\mathscr{A}(S)&=\int_{-1}^0\left(\frac{x^3}{4}-x\right)dx+\int_0^2\left(x-\frac{x^3}{4}\right)dx=\left(\frac{x^4}{16}-\frac{x^2}{2}\right)\Big|_{-1}^0+\left(\frac{x^2}{2}-\frac{x^4}{16}\right)\Big|_0^2\\&=2-1-\frac{1}{16}+\frac{1}{2}=\frac{23}{16}.\end{aligned}$$

■

## 11.2 Area of flat figures in polar coordinates

Suppose we have a curve given as $r=f(\theta)$, where $r$ is the distance to the origin and $\theta$ the angle with the positive X axis. These two variables are known as polar coordinates, and its relation to the cartesian coordinates is given by the transformation (see Figure 11.2(a))

$$x=r\cos\theta, \qquad y=r\sin\theta. \tag{11.3}$$

Figure 11.2(b) illustrates a curve $r=f(\theta)$ expressed in polar coordinates.

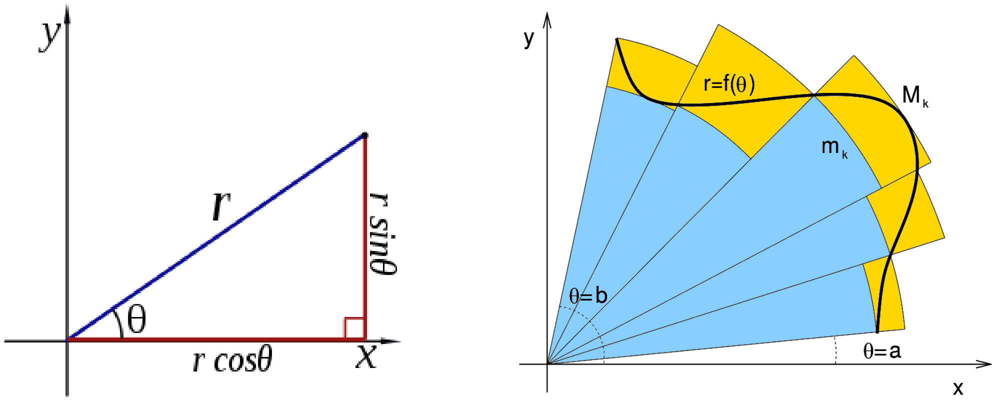
The problem we face now is that of calculating the area of the figure formed by the curve $r=f(\theta)$ and the radii at angles $\theta=a$ and $\theta=b$, i.e., the figure $S=\big\{(r,\theta)\,:\,a\leqslant\theta\leqslant b,\ 0\leqslant r\leqslant f(\theta)\big\}$. In order to achieve that we can introduce the analogue of upper and lowe sums. If we introduce the partition $P=\{\theta_0,\theta_1,\dots,\theta_n\}$, where $a=\theta_0<\theta_1<\cdots<\theta_{n-1}<\theta_n=b$ and define

$$m_i=\inf_{\theta_{i-1}\leqslant\theta\leqslant\theta_i} f(\theta), \qquad M_i=\sup_{\theta_{i-1}\leqslant\theta\leqslant\theta_i} f(\theta), \tag{11.4}$$

then $\mathscr{A}(S)$, the area of $S$, should be bounded as

$$\sum_{i=1}^n \frac{1}{2}m_i^2(\theta_i-\theta_{i-1})\leqslant\mathscr{A}(S)\leqslant\sum_{i=1}^n\frac{1}{2}M_i^2(\theta_i-\theta_{i-1}). \tag{11.5}$$

If $f^2$ is an integrable function, this is equivalent to the Riemann definition of the integral. Thus, in the limit we will find




(a) (b)

Figure 11.2: (a) Polar coordinates. (b) A curve expressed in polar coordinates $r = f(\theta)$ and its associated upper and lower sums construction. Here rectangles are replaced by circular sectors.

$$\mathscr{A}(S) = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta. \tag{11.6}$$

■ **Example 11.3** Let us calculate the area enclosed by the curve $(x^2+y^2)^3 = y^2$ —represented in Figure 11.3.

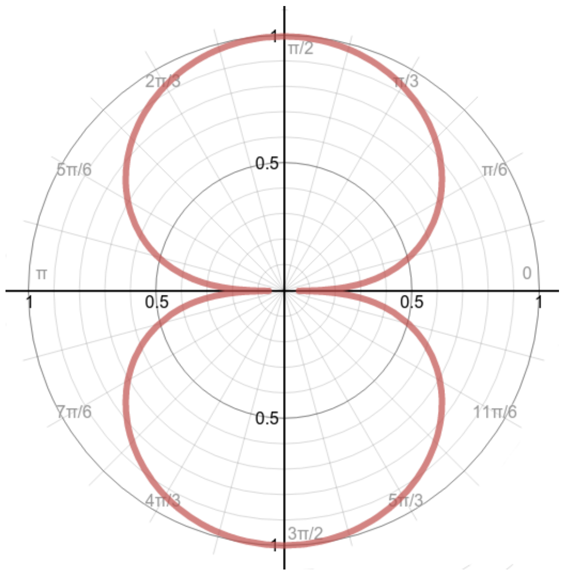



Figure 11.3: Curve $(x^2+y^2)^3 = y^2$.

To begin with, we will rewrite the curve in polar coordinates as

$$(r^2)^3 = r^2 \sin^2\theta \qquad \Rightarrow \qquad r^4 = \sin^2\theta \qquad \Rightarrow \qquad r = \sqrt{|\sin\theta|} = f(\theta), \quad 0 \leqslant \theta \leqslant 2\pi.$$

Accordingly

$$\mathscr{A}(S) = \frac{1}{2}\int_0^{2\pi} |\sin\theta|\, d\theta = \int_0^{\pi} \sin\theta\, d\theta = (-\cos\theta)\Big|_0^{\pi} = 2.$$

■

## 11.3 Volumes

### 11.3.1 Solids of a given section

Suppose we have a solid $S$ whose sections parallel to certain plane at a distance $u$ from that plane are given by the function $a_S(u)$. Suppose further that the solid spans all distances $a \leqslant u \leqslant b$. Then the volume of $S$ will be given by

$$\mathscr{V}(S) = \int_a^b a_S(u)\,du. \tag{11.7}$$

The proof of this result —first used by Pappus of Alexandria in the 4th century— follows the same spirit as Riemann's upper and lower sums construction. Suppose we make a partition $P = \{a = u_0, u_1, \ldots, u_{n-1}, u_n = b\}$ of the segment $[a,b]$. Any sum

$$\sum_{i=1}^{n} a_S(c_i)(u_i - u_{i-1}), \qquad u_{i-1} \leqslant c_i \leqslant u_i,$$

is a Riemann sum of the function $a_S(u)$ which represent the sum of the volumes of a stack of parallelepipeds that provides an estimate of the volume of the solid $S$. Thus we obtain formula (11.7) as long as $a_S(u)$ is an integrable function.

■ **Example 11.4** As an example let us calculate the volume of a square pyramid of base side $l$ and height $h$. If we take the basal plane as the reference plane, sections parallel to the base at height $u$ are squares of side $x$ (see Figure 11.4). The value of $x$ can be determined by triangle similarity as

$$\frac{h}{l/2} = \frac{u}{(l-x)/2} \qquad \Rightarrow \qquad x = l\left(1 - \frac{u}{h}\right).$$

Accordingly

$$a_S(u) = l^2\left(1 - \frac{u}{h}\right)^2$$

and therefore

$$\mathscr{V}(s) = \int_0^h l^2\left(1 - \frac{u}{h}\right)^2 du = l^2\left[-\frac{h}{3}\left(1 - \frac{u}{h}\right)^3\right]\Bigg|_0^h = \frac{l^2 h}{3}.$$

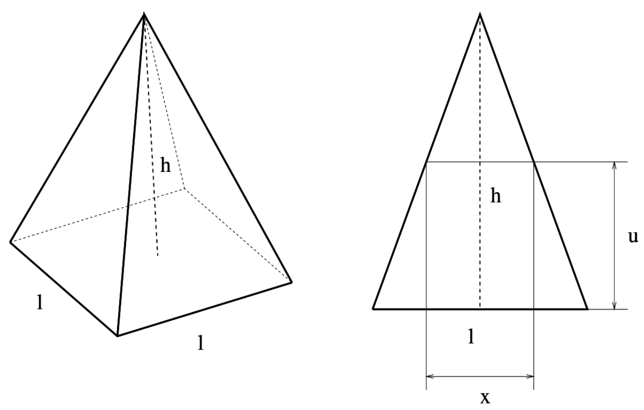



Figure 11.4: Volume of a square pyramid of base side $l$ and height $h$ calculated through its sections.

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### 11.3.2 Solids of revolution

As a special application of formula (11.7) we can obtain formulas for solids of revolution —i.e., solids generated by the turn of a plane figure around the X or the Y axis.

#### Around the X axis

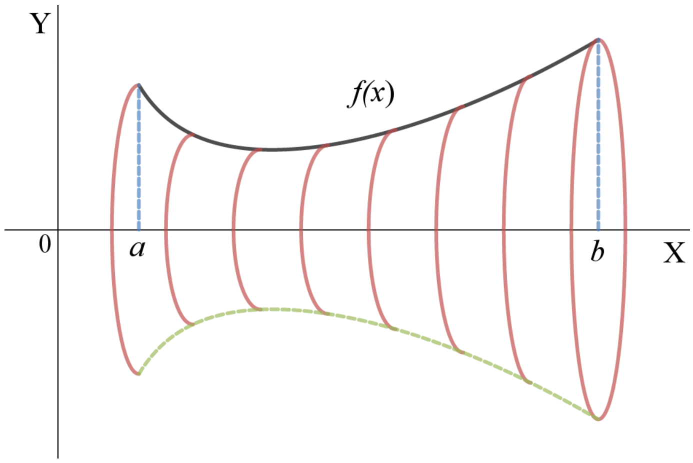


Figure 11.5: Solid of revolution generated by the curve $y = f(x)$ with $a \leqslant x \leqslant b$ as it revolves around the X axis.

As illustrated in Figure 11.5, the portion of the curve $y = f(x)$ within the interval $[a,b]$ generates, as it revolves around the X axis, the surface of a solid of revolution $S$. Sections of this solid perpendicular to the X axis are disks. If the section is taken at $a \leqslant x \leqslant b$, then the radius of the disk is $f(x)$. Therefore $a_S(x) = \pi f(x)^2$ and consequently

$$\mathscr{V}(S) = \pi \int_a^b f(x)^2\,dx. \tag{11.8}$$

■ **Example 11.5** To determine the volume of a sphere of radius $a$ we can take as a solid of revolution of the curve $f(x) = \sqrt{a^2 - x^2}$ within the interval $[-a,a]$. Thus,

$$\mathscr{V}(S) = \pi \int_{-a}^{a} (a^2 - x^2)\,dx = \pi \left( a^2 x - \frac{x^3}{3} \right) \Bigg|_{-a}^{a} = 2\pi \left( a^3 - \frac{a^3}{3} \right) = \frac{4\pi}{3} a^3.$$

■

#### Around the Y axis

In the case that the flat figure between the curve $y = f(x)$ and the X axis delimited by the interval $[a,b]$ revolves around the $Y$ axis, we obtain a solid of revolution $S$ as that of Figure 11.6. To calculate its volume we need to adapt Pappus's construction a little bit, because each interval of a partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$, along with the corresponding heights $f(c_i)$ (where $x_{i-1} \leqslant c_i \leqslant x_i$), generates a hollow cylinder (a tube). The volume of that cylinder is the difference between that of the outer cylinder $\pi f(c_i) x_i^2$ and that of the inner cylinder $\pi f(c_i) x_{i-1}^2$. Thus, the Riemann sum

$$\sum_{i=1}^{n} \pi (x_i^2 - x_{i-1}^2) f(c_i) \tag{11.9}$$

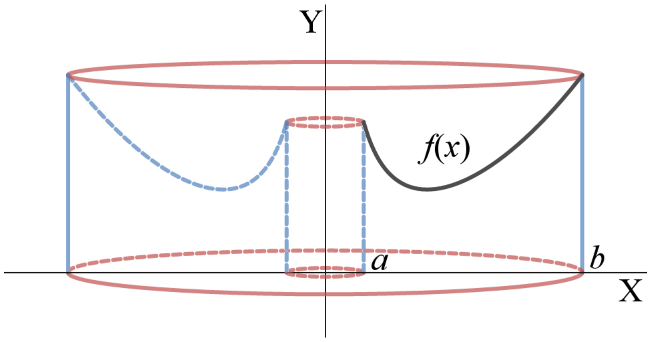


Figure 11.6: Solid of revolution generated by the curve $y = f(x)$ with $a \leqslant x \leqslant b$ as it revolves around the Y axis.

provides an estimate of the volume of the solid $S$. This is valid for any $c_i$, but if we choose $c_i = (x_i + x_{i-1})/2$ we can rewrite in a much better way as

$$\sum_{i=1}^{n} 2\pi c_i f(c_i)(x_i - x_{i-1}). \tag{11.10}$$

This is a Riemann sum of the function $g(x) = 2\pi x f(x)$, so if $g$ is integrable in $[a,b]$ then

$$\mathscr{V}(S) = 2\pi \int_a^b x f(x)\, dx. \tag{11.11}$$

■ **Example 11.6** Let us calculate the volume of a doughnut (a *torus* in mathematical parlance), the solid represented in Figure 11.7. This volume will be twice the volume generated by the half-disk delimited by the function $f(x) = \sqrt{a^2 - (x-R)^2}$ and the X axis within the interval $[R-a, R+a]$, as it revolves around the Y axis. Hence

$$\mathscr{V}(S) = 4\pi \int_{R-a}^{R+a} x\sqrt{a^2 - (x-R)^2}\, dx$$

With the change of variable $x = R + a\sin\theta$ (thus $x' = a\cos\theta$) we transform the integral into

$$\mathscr{V}(S) = 4\pi \int_{-\pi/2}^{\pi/2} (R + a\sin\theta)a^2\cos^2\theta\, d\theta$$

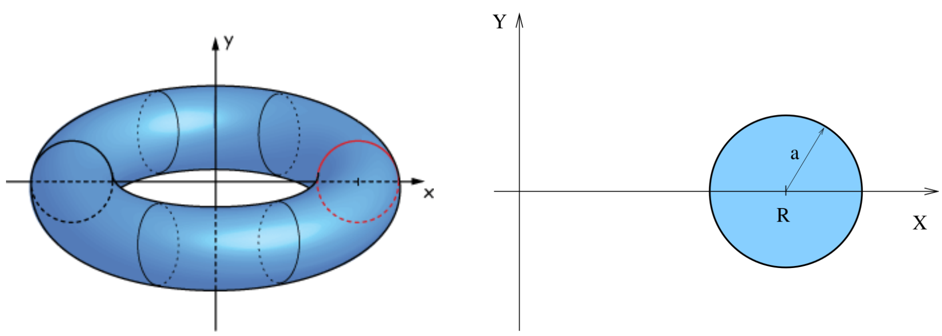


Figure 11.7: Solid of revolution generated by the curve $(x-R)^2 + y^2 = a^2$ as it revolves around the Y axis. This solid —actually a doughnut— is called in mathematics *torus*.

Now, $\sin\theta\cos^2\theta$ is an odd function, so

$$\int_{-\pi/2}^{\pi/2} \sin\theta\cos^2\theta\, d\theta = 0$$

and therefore

$$\mathscr{V}(S) = 4\pi Ra^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2\pi Ra^2 \int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\, d\theta = 2\pi Ra^2 \Big(\pi + \underbrace{\frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2}}_{=0}\Big)$$

$$= (2\pi R)(\pi a^2).$$

■

## 11.4 Length of curves

Consider a parametric curve $C = \{\mathbf{r}(t) \in \mathbb{R}^n : a \leqslant t \leqslant b\}$, and the partition of the interval $[a,b]$ defined by $P = \{a = t_0, t_1, \ldots, t_{n-1}, t_n = b\}$. If we join with straight segments the point $\mathbf{r}(t_0)$ with $\mathbf{r}(t_1)$, the point $\mathbf{r}(t_1)$ with $\mathbf{r}(t_2)$, and so on and so forth up to $\mathbf{r}(t_{n-1})$ with $\mathbf{r}(t_n)$, we obtain a polygonal curve $\Pi(P)$ that approximates the curve $C$ (see Figure 11.8). The length of $\Pi(P)$ is easy to calculate:

$$\mathscr{L}\big(\Pi(P)\big) = \sum_{i=1}^{n} \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|. \tag{11.12}$$

If we multiply and divide each term of this sum by the length of the interval of the parameter we obtain

$$\mathscr{L}\big(\Pi(P)\big) = \sum_{i=1}^{n} \left\| \frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \right\| (t_i - t_{i-1}). \tag{11.13}$$

As we refine more and more the partition

$$\frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \xrightarrow[t_i - t_{i-1} \to 0]{} \mathbf{r}'(t_i)$$

and $\mathscr{L}\big(\Pi(P)\big)$ gets closer and closer to a Riemann sum of the function $\|\mathbf{r}'(t)\|$. Thus, if this function is integrable,

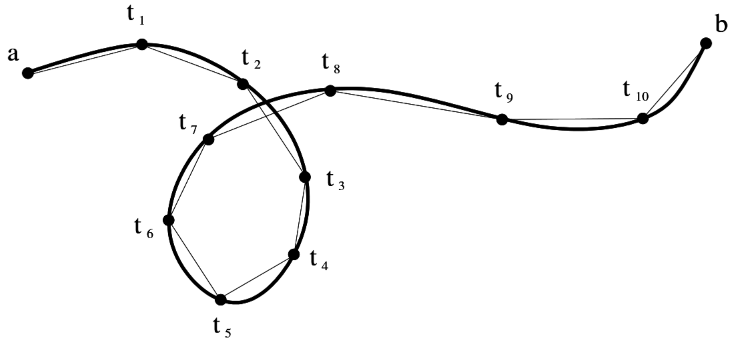


Figure 11.8: A curve along with its polygonal curve associated to a partition.

$$\mathscr{L}(C) = \int_a^b \|r'(t)\|\,dt. \tag{11.14}$$

■ **Example 11.7** A circumference of radius $a$ is a plane curve $C$ given by the parametrisation $\mathbf{r}(t) = (a\cos t, a\sin t)$, where $0 \leqslant t \leqslant 2\pi$. Thus, its length can be obtained as

$$\mathscr{L}(C) = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = \int_0^{2\pi} a\,dt = 2\pi a.$$

■

## Problems

**Problem 11.1** Calculate the area delimited by the following curves:

(i) $y = x^2$, $y = (x-2)^2$, $y = (2-x)/6$;

(ii) $x^2 + y^2 = 1$, $x^2 + y^2 = 2x$;

(iii) $y = \dfrac{1-x}{1+x}$, $y = \dfrac{2-x}{1+x}$, $y = 0$, $y = 1$;

(iv) one loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

**Problem 11.2** Determine the area between the curve $f(x) = \dfrac{x(x^2-1)}{(x^2+1)^{3/2}}$ and the X axis.

**Problem 11.3** Calculate the area delimited by the following curves:

(i) $r = a\theta$ (Archimedes's spiral), $0 \leqslant \theta \leqslant 2\pi$, and the segment $\{(x,0) : 0 \leqslant x \leqslant 2\pi a\}$;

(ii) a petal of the three-petal rose $r = a\cos 3\theta$, $-\pi/6 \leqslant \theta \leqslant \pi/6$;

(iii) half a *lemniscata* $r = a\sqrt{\cos 2\theta}$, $-\pi/4 \leqslant \theta \leqslant \pi/4$.

**Problem 11.4** Let $A$ the plane figure limited by the curves $y = x^2$ and $y = \sqrt{x}$. Determine:

(a) the area of $A$;

(b) the volume of the solid generated when $A$ revolves around the X axis.

**Problem 11.5** Compute the volume of the solids generated when the following sets revolve around the X asis:

(i) $0 \leqslant y \leqslant 1 + \sin x$, $0 \leqslant x \leqslant 2\pi$;

(ii) $R^2 \leqslant x^2 + y^2 \leqslant 4R^2$;

(iii) plane figure delimited by the curves $y = \sin x$ and $y = x$ with $0 \leqslant x \leqslant \pi$.

**Problem 11.6** Compute the volume of the following solids:

(i) the solid generated when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ revolves around the X axis;

(ii) same thing around the Y axis;

(iii) the solid whose base is the ellipse above and whose sections perpendicular to the X axis are isosceles triangles of height 2.

**Problem 11.7**

(a) Calculate the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(b) Calculate the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

(c) Check the result of Problem 11.6 (i) and (ii) as particular cases of the previous result.

HINT: Notice that intersecting the ellipsoid by planes parallel to the coordinate planes ($x = 0$, $y = 0$, or $z = 0$) we obtain ellipses.

**Problem 11.8** Calculate the length of the following curves:

(i) catenary: $y = e^{x/2} + e^{-x/2}$, $0 \leqslant x \leqslant 2$;

(ii) cycloid: $x(t) = a(t - \sin t)$, $y(t) = a(1 - \cos t)$, $0 \leqslant t \leqslant 2\pi$;

(iii) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4$;

(iv) tractrix: $y = a\log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}$, $a/2 \leqslant x \leqslant a$;

(v) cardioid: $r = 1 + \cos\theta$, $0 \leqslant \theta \leqslant 2\pi$.