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Calculus I

Pablo Catalán Fernández y José A. Cuesta Ruiz

Unit 2. Real Functions

2. Real Functions

2.1 Definition and basic concepts

Formally, a **real function** is a map from a set $A \subset \mathbb{R}$ to $\mathbb{R}$. In practical terms, it is a “rule” that “assigns” one —and only one!— real number to each element $x \in A$. A basic notation for a function is

$$\begin{aligned} f: A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) \end{aligned} \tag{2.1}$$

Functions are also referred to as *maps* or *mappings*. They are usually denoted $y = f(x)$, where f represents the rule that assigns y to x.

■ Example 2.1

(a) $y = x^2$ represents the rule $f(x) = x^2$ that maps each number x to its square.
(b) $y = |x|$ represents the rule $f(x) = |x|$ that maps each number x to its absolute value.
(c) The function

$$f(x) = \begin{cases} x^2 & x \leqslant 2, \\ x^3 - 3 & x > 2, \end{cases}$$

maps all real numbers smaller than or equal to 2 to their square, and those larger than 2 to their cube minus 3.
(d) The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

maps all rational numbers to 1 and all irrational numbers to 0.

■

The **domain** of a function is the set A. This domain is maximal if the function cannot be defined for numbers $x \notin A$.

The **image** or **range** of a function is the set $f(A) \equiv \{f(x) : x \in A\}$.

Likewise, we call *image of the set* $C \subset A$ to the set $f(C) \equiv \{f(x) : x \in C\}$.

We call *inverse image of a set* $B \subset \mathbb{R}$ to the set $f^{-1}(B) \equiv \{x \in A : f(x) \in B\}$. Note that $f^{-1}(B) \subset A$.

The **graph** of a function $f(x)$ is the subset of $\mathbb{R}^2$ defined by the points $\{(x, f(x)) : x \in A\}$. Plotting this set is how we represent functions.

A function is **injective,** or **one-to-one,** if for every pair of number $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$. If a function is injective, the equation $y = f(x)$ has either no solution or a unique solution.

A function is **surjective,** or **onto,** if $f(A) = \mathbb{R}$. If a function is surjective, the equation $y = f(x)$ always has at least one solution.

A function is **bijective** if it is both injective and surjective. If a function is bijective, the equation $y = f(x)$ always has one, and only one, solution for each $y \in \mathbb{R}$.

A function is **periodic** if there exists $c > 0$ such that $f(x+c) = f(x)$. The smallest such c is referred to as the *period* of the function.

A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

A function is **bounded** if there exists $M > 0$ such that $|f(x)| \leqslant M$ for all x in its domain.

A function is **monotonically increasing** if for every x, y in its domain such that $x < y$ it satisfies $f(x) \leqslant f(y)$, and is **monotonically decreasing** if $f(x) \geqslant f(y)$. We say it is **monotonic strictly increasing/decreasing** if inequalities are strict. (Note that a constant is both monotonically increasing and decreasing, but not strictly.)

■ **Example 2.2**

(a) The domain of $f(x) = x^2$ is $\mathbb{R}$ and its image is $f(\mathbb{R}) = [0, \infty)$. This function is not injective because x and $-x$ have the same square. It is not surjective either because $f(\mathbb{R}) \neq \mathbb{R}$. The inverse image of the interval $[4,9]$ is $f^{-1}([4,9]) = [-3,-2] \cup [2,3]$.

(b) The domain of $f(x) = \log x$ is $(0, \infty)$ and its image is $\mathbb{R}$. It is injective because two different numbers have different logarithms. It is also surjective because any number y is always the logarithm of a number, namely e^y. So it is bijective.

(c) $F(x) = e^x - e^{-x}$ is an odd function because $f(-x) = e^{-x} - e^x = -f(x)$.

(d) $f(x) = \cos x$ is even because $\cos(-x) = \cos x$.

(e) $f(x) = \sin^2 x$ is periodic of period π because $\sin^2(x+\pi) = \sin^2 x$.

■

2.2 Elementary functions

There is a wide range of elementary functions that we will work with. They include polynomials, rational functions, trigonometric functions, the exponential and the logarithm.

2.2.1 Polynomials

These are functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{2.2}$$

where $a_k \in \mathbb{R}$ for all $k = 0, 1, \ldots, n$. The largest power, n, is called the *degree* of the polynomial. Constants are polynomials of degree 0. Given the operations that define them, the domain of any polynomial is $\mathbb{R}$.

2.2.2 Rational functions

They are defined as quotients of two polynomials, namely

$$f(x) = \frac{P_n(x)}{Q_m(x)}. \tag{2.3}$$

The domain of both polynomials is $\mathbb{R}$, but $Q_m(x)$ may be zero at some points, where the quotient will thus not be defined. Hence the domain of $f(x)$ is $\{x \in \mathbb{R} : Q_m(x) \neq 0\}$.

2.2.3 Trigonometric functions

The two basic trigonometric functions are the sine ($\sin x$) and the cosine ($\cos x$). In terms of them we can define also the tangent and cotangent:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}. \tag{2.4}$$

The geometric definition of these functions, based on the unit circle, is described in Figure 2.1.

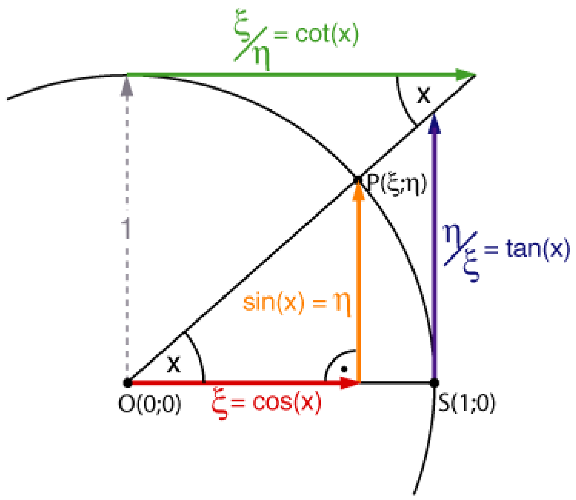

Figure 2.1: Geometric definition of $\sin x$, $\cos x$, $\tan x$, and $\cot x$.

There are two more trigonometric functions, although less common than the previous one, namely the secant ($\sec x$) and the cosecant ($\csc x$):

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}. \tag{2.5}$$

The graphs of $\sin x$ and $\cos x$ are plotted in Figure 2.2. Those of $\tan x$ and $\cot x$ in Figure 2.3.

Trigonometric identities	
$\cos^2 x + \sin^2 x = 1$	$1 + \cot^2 x = \csc^2 x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$
$\cos 2x = \cos^2 x - \sin^2 x$	$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
$\sin 2x = 2 \sin x \cos x$	$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
$1 + \tan^2 x = \sec^2 x$	$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Table 2.1: Some important trigonometric identities.

Given their geometric definitions, all these functions are related by geometric identities. The main one are listed in Table 2.1.

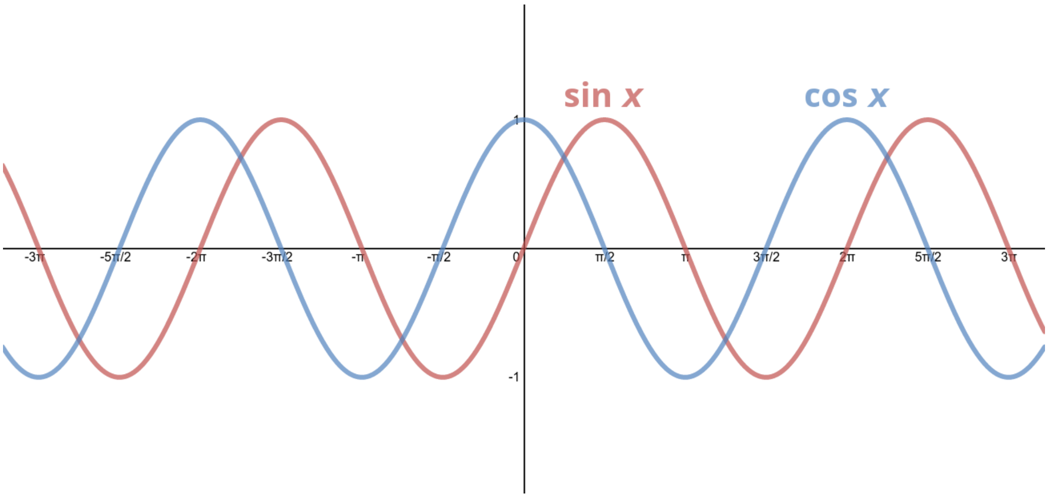

Figure 2.2: Plot of $\sin x$ and $\cos x$.

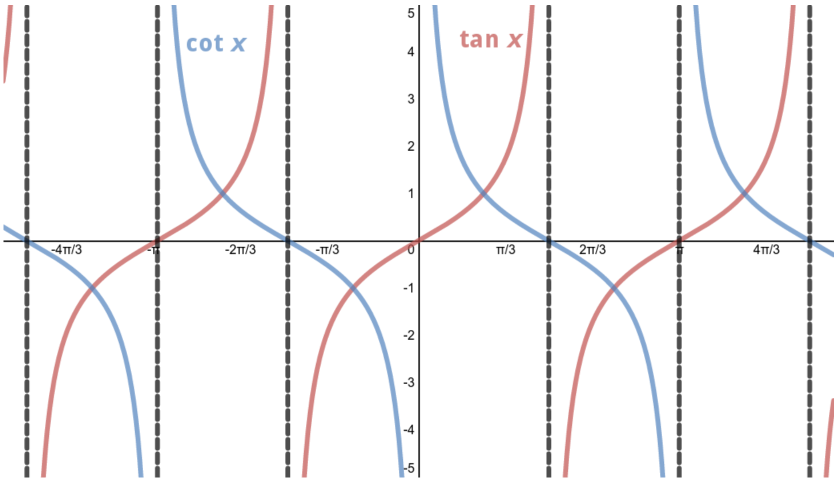

Figure 2.3: Plot of $\tan x$ and $\cot x$.

2.2.4 Exponential

This is the function defined as $f(x) = e^x$. The constant e appearing in this definition is the irrational number introduced by Euler

$$e = 2.71828182845904523536028747135266249775724709369995957496 6\ldots$$

We will see a proper definition of this constant later on. Apart from that, the definition of the exponential involves raising a real number to a real power. This requires some clarifications.

Integer powers of real numbers are easily defined through the concept of repeated product. Thus $e^3 = e \cdot e \cdot e$. With this definition, for any $n, m \in \mathbb{N}$ it is straightforward that

$$e^{n+m} = e^n e^m, \tag{2.6}$$

from which it follows

$$(e^m)^n = \underbrace{e^m \cdot e^m \cdots e^m}_{n \text{ times}} = e^{m+m+\cdots+m} = e^{nm}. \tag{2.7}$$

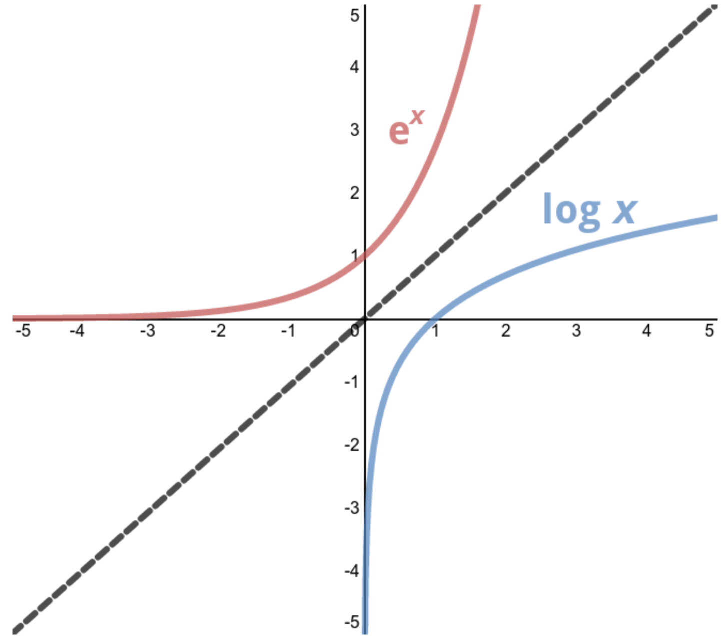

Figure 2.4: Plot of e^x and $\log x$.

We will take these formulas as a basic definition. Extending them will provide meaning to powers other than natural numbers. For instance, applying (2.6),

$$e^{n-m}e^m = e^{n-m+m} = e^n \quad \Rightarrow \quad e^{n-m} = \frac{e^n}{e^m}.$$

But extending (2.6) means assuming $e^{n-m} = e^n e^{-m}$. Cancelling a factor e^n in both sides leads to

$$e^{-m} = \frac{1}{e^m},$$

which provides a meaning to negative powers. And from this definition it follows

$$e^0 = e^{n-n} = \frac{e^n}{e^n} = 1.$$

As for fractional powers, equation (2.7) implies

$$(e^{1/n})^n = e^{n/n} = e \quad \Rightarrow \quad e^{1/n} = \sqrt[n]{e}.$$

Thus, $e^{m/n} = \sqrt[n]{e^m}$. This extension of the basic multiplicative rule provides a definition of the exponential valid for all rational powers. It only remains to define it for irrational powers. But irrational numbers can be approximated as much as we like by rational numbers. If fact, as we have seen, irrational numbers can be bracketed by sequences of rational approximants; i.e., if x is an irrational number, we can find two sequences of rational numbers such that

$$p_1 < p_2 < p_3 < \cdots < p_n < \cdots < x < \cdots < q_n < \cdots < q_3 < q_2 < q_1.$$

Thus we can define e^x as the number bracketed by

$$e^{p_1} < e^{p_2} < e^{p_3} < \cdots < e^{p_n} < \cdots < e^x < \cdots < e^{q_n} < \cdots < e^{p_3} < e^{p_2} < e^{p_1}.$$

Using this definition we can summarise the properties of the exponential as follows:

1. Its domain is $\mathbb{R}$.
2. $e^x > 0$ for all $x \in \mathbb{R}$.
3. It is monotonic strictly increasing —hence injective.
4. $e^0 = 1$.
5. $(e^x)^a = e^{ax}$ for any $a \in \mathbb{R}$.
6. $e^{x+y} = e^x e^y$.
7. $e^{-x} = 1/e^x$.

A plot of the exponential function is shown in Figure 2.4.

Some important functions defined in terms of exponentials define what is known as the *hyperbolic trigonometry*. The main one are the hyperbolic cosine ($\cosh x$) and sine ($\sinh x$), defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}. \tag{2.8}$$

Their plots are shown in Figure 2.5. It can be seen that $\cosh x$ is even whereas $\sinh x$ is odd.

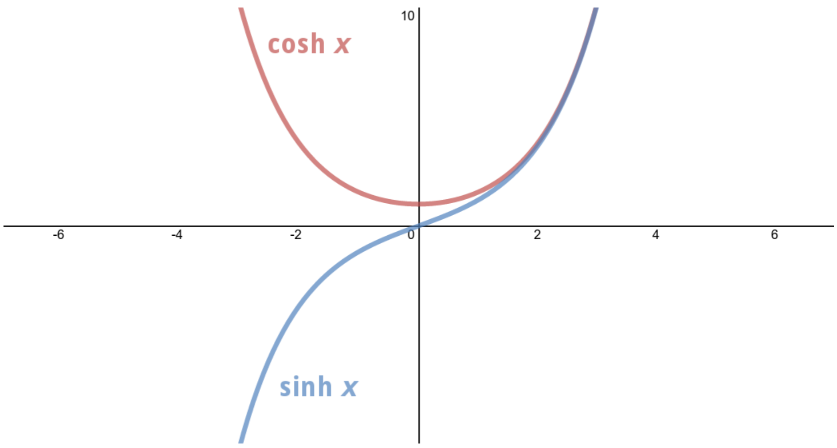

Figure 2.5: Plot of $\cosh x$ and $\sinh x$.

Hyperbolic tangent ($\tanh x$) and cotangent ($\coth x$) can also be defined (see Figure 2.6 for their plots):

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}, \tag{2.9}$$

and similarly $\operatorname{sech} x = 1/\cosh x$ and $\operatorname{csch} x = 1/\sinh x$.

There is a list of identities relating these functions similar to that of the ordinary trigonometry, as illustrated in Table 2.2.

2.2.5 Logarithm

This is the inverse of the exponential. If $y = \log x$ it means that $x = e^y$. Its plot can be seen in Figure 2.4 to mirror that of the exponential with respect to the line $y = x$.

> **R** Along these notes, whenever we write $x = \log y$ we mean that x is the solution of the equation $e^x = y$, in other words, log of a number is the exponent to which we need to rise e in order to obtain that number. In particular $\log 1 = 0$ and $\log e = 1$.

The main properties of the logarithm (derived from those of the exponential) are the following:

1. Its domain is $(0, \infty)$.

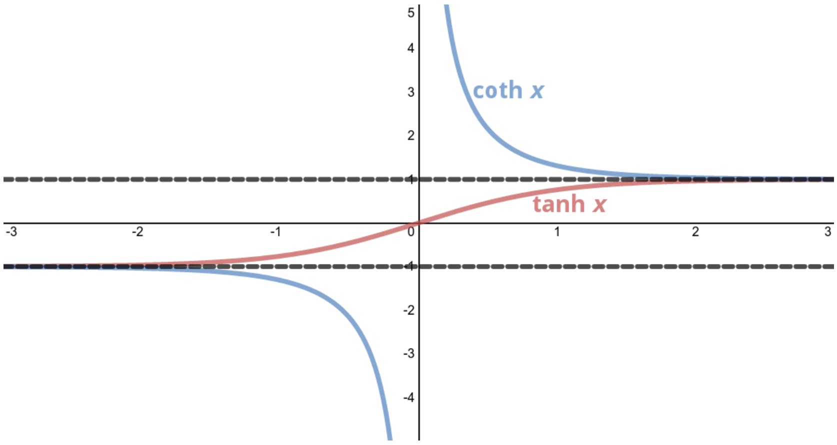

Figure 2.6: Plot of $\tanh x$ and $\coth x$.

Hyperbolic trigonometric identities	
$\cosh^2 x - \sinh^2 x = 1$	$\coth^2 x - 1 = \operatorname{csch}^2 x$
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$	$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cosh x \cosh y = \frac{1}{2}[\cosh(x+y) + \cosh(x-y)]$
$\sinh 2x = 2\sinh x \cosh x$	$\sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh x \cosh y = \frac{1}{2}[\sinh(x+y) + \sinh(x-y)]$

Table 2.2: Some important trigonometric identities.

2. Its image is $\mathbb{R}$ —hence it is surjective.
3. It is monotonic strictly increasing —hence injective.
4. $\log 1 = 0$.
5. $\log(x^a) = a \log x$.
6. $\log(xy) = \log x + \log y$.
7. $\log(x/y) = \log x - \log y$.

2.3 Operations with functions

2.3.1 Algebraic operations

Let $A, B \subset \mathbb{R}$ and consider the two real functions

$$\begin{aligned} f : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) \end{aligned} \qquad \begin{aligned} g : B &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = g(x) \end{aligned} \tag{2.10}$$

With these two functions we can perform the following algebraic operations:

(i) **Addition:** If $C = A \cap B$ —where both functions are defined—,

$$\begin{aligned} f + g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) + g(x) \end{aligned} \tag{2.11}$$

(ii) **Product:** If $C = A \cap B$,

$$\begin{aligned} fg : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x)g(x) \end{aligned} \tag{2.12}$$

(iii) **Quotient:** If $C = A \cap B'$, where $B' \equiv \{x \in B \,:\, g(x) \neq 0\}$,

$$\begin{aligned} f/g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x)/g(x) \end{aligned} \tag{2.13}$$

For example, a polynomial is actually a sum of monomials, each a different function; or a rational function is the quotient of two polynomials.

2.3.2 Compositions

A more involved operation is the composition of two functions. It is defined as

$$\begin{aligned} f \circ g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f\big(g(x)\big) \end{aligned} \tag{2.14}$$

The problem is to find the domain of this function, given the domains A and B of the composed functions. For $f \circ g$ to be defined x must belong to B, for $g(x)$ to be well defined, so $C \subset B$. But in order to evaluate $f\big(g(x)\big)$, the number $g(x) \in A$. Therefore

$$C = \{x \in B \,:\, g(x) \in A\}. \tag{2.15}$$

Even if A and B are simple sets, C may be much more involved.

■ **Example 2.3** Consider the functions $f(x) = 1/(x-1)$ and $g(x) = \sin x$. Clearly $A = \mathbb{R} - \{1\}$ and $B = \mathbb{R}$, two very simple sets. However, the domain of their composition $f \circ g$ is the domain of the function

$$(f \circ g)(x) = \frac{1}{\sin x - 1},$$

i.e., $\mathbb{R}$ excluding those values for which $\sin x = 1$ (because the denominator vanishes). This set is

$$C = \mathbb{R} - \left\{ \left(2n + \frac{1}{2}\right)\pi \,:\, n \in \mathbb{Z} \right\}.$$

■

Composition is a noncommutative operation, i.e., $f \circ g \neq g \circ f$. In the example above, $(f \circ g)(x) = 1/(\sin x - 1)$ is very different from $(g \circ f)(x) = \sin\left(\frac{1}{x-1}\right)$.

It is, however, associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$. We can thus define multiple compositions, like $f \circ g \circ h \circ w = f(g(h(w(x))))$, without ambiguity.

2.3.3 Inverses

We can introduce the identity function $\mathrm{Id}(x) = x$. Given a function $f : A \longrightarrow \mathbb{R}$, its **inverse** would be a function $f^{-1} : f(A) \longrightarrow \mathbb{R}$ such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{Id}$. The idea is that if f maps x to y, its inverse f^{-1} maps y back to x.

Not all functions have an inverse that is defined all over their image $f(A)$. For an inverse to exist the equation $x = f(y)$, for a given $x \in f(A)$, must have a unique solution;[1] in other words, f

[1] It already has at least one solution because $x \in f(A)$.

must be injective. Monotonic strictly increasing or decreasing functions are injective. This is why the exponential has an inverse —the logarithm.

For those functions that are not injective in their domain A, we might be able to define several inverses by constraining the domain to any subset where they are made injective. Thus, noninjective functions may have several inverses.

■ **Example 2.4** Let $f(x) = x^2$. Its domain is $\mathbb{R}$, but this function is not injective in its domain. However, we can constraint the domain to be $[0,\infty)$. In that case $f(x)$ is injective and we can obtain the inverse function by finding the unique solution of the equation $x = f(y) = y^2$, where $0 \leqslant y$. Clearly this solution is $y = \sqrt{x}$, therefore, within $[0,\infty)$, the inverse of f is $f^{-1}(x) = \sqrt{x}$.

Note that we might alternatively chosen the domain to be $(-\infty,0]$, where the function f is again injective. However now the solution of $x = y^2$ with $y \leqslant 0$ is $y = -\sqrt{x}$. So another inverse of f is $f^{-1}(x) = -\sqrt{x}$. ■

■ **Example 2.5** Periodic functions are clearly not injective. Take $\sin x$, for instance. An interval where it is injective is $[-\pi/2,\pi/2]$. The inverse of this function within this interval is usually called the *arc sine:* $\sin^{-1} x = \arcsin x$. But we might have taken the interval $[\pi/2,3\pi/2]$, for instance. In that case the inverse would be different: $\sin^{-1} x = \pi - \arcsin x$. Or in the interval $[3\pi/2,5\pi/2]$ the inverse would be $\sin^{-1} x = 2\pi + \arcsin x$.

Similarly, $\arccos x = \cos^{-1} x$ when the domain of $\cos x$ is taken to be $[0,\pi]$, and $\arctan x = \tan^{-1} x$ when the domain of $\tan x$ is taken to be $(-\pi/2,\pi/2)$.

Note that $\arccos x$ (or $\operatorname{arccot} x$ for that matter) is redundant, because

$$\arccos x = \frac{\pi}{2} - \arcsin x, \qquad \operatorname{arccot} x = \frac{\pi}{2} - \arctan x.$$

■

The graph of $f^{-1}(x)$ can be obtained from that of $f(x)$ as the mirror image with respect to the line $y = x$ (see Figure 2.4).

R BEWARE!! Never confuse $f^{-1}(x)$ with $f(x)^{-1} = 1/f(x)$. In the case $f(x) = \sin x$, is inverse $\sin^{-1} x = \arcsin x$, whereas $(\sin x)^{-1} = \csc x$.

Exercise 2.1 Argue that $\sinh x$ has a unique inverse over $\mathbb{R}$ and that it can be obtained as

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right). \tag{2.16}$$

The function $\cosh x$ has two inverses (why?) that can be obtained as

$$\cosh^{-1} x = \pm\log\left(x + \sqrt{x^2-1}\right). \tag{2.17}$$

HINT: For $\sinh x$, solve $2x = e^y - e^{-y}$ by transforming it into $e^{2y} - 2xe^y - 1 = 0$ and reading it as a quadratic equation in e^y. Use a similar procedure for $\cosh x$.

Find an expression for $\tanh^{-1} x$. ■

Problems

Problem 2.1 Determine the domain of the following functions:

(i) $f(x) = \dfrac{1}{x^2-5x+6}$;

(ii) $f(x) = \sqrt{1-x^2}+\sqrt{x^2-1}$;

(iii) $f(x) = \dfrac{1}{x-\sqrt{1-x^2}}$;

(iv) $f(x) = \sqrt{1-\sqrt{4-x^2}}$;

(v) $f(x) = \dfrac{1}{1-\log x}$;

(vi) $f(x) = \log(x-x^2)$;

(vii) $f(x) = \dfrac{\sqrt{5-x}}{\log x}$;

(viii) $f(x) = \arcsin(\log x)$.

Problem 2.2

(a) If f and g are both odd functions, what are $f+g$, fg, and $f\circ g$?

(b) And what are the same functions if now f is even and g is odd?

Problem 2.3 Check whether the following functions are even or odd:

(i) $f(x) = \dfrac{x}{x^2+1}$;

(ii) $f(x) = \dfrac{x^2-x}{x^2+1}$;

(iii) $f(x) = \dfrac{\sin x}{x}$;

(iv) $f(x) = \cos(x^3)\sin(x^2)e^{-x^4}$;

(v) $f(x) = \dfrac{1}{\sqrt{x^2+1}-x}$;

(vi) $f(x) = \log\left(\sqrt{x^2+1}-x\right)$.

Problem 2.4 For which numbers $a,b,c,d\in\mathbb{R}$ the function $f(x)=\dfrac{ax+b}{cx+d}$ is its own inverse (i.e., $f\circ f=\mathrm{Id}$) in the domain of f?

Problem 2.5 Check that the function $f(x)=\dfrac{x+3}{1+2x}$ is bijective and maps its domain $\mathbb{R}-\{-1/2\}$ to $\mathbb{R}-\{1/2\}$.

Problem 2.6

(a) Determine which of these functions are injective. For those that are obtain their inverse. For those that are not, find two points with the same image.

(i) $f(x)=7x-4$;

(ii) $f(x)=\sin(7x-4)$;

(iii) $f(x)=(x+1)^3+2$;

(iv) $f(x)=\dfrac{x+2}{x+1}$;

(v) $f(x)=x^2-3x+2$;

(vi) $f(x)=\dfrac{x}{x^2+1}$;

(vii) $f(x)=e^{-x}$;

(viii) $f(x)=\log(x+1)$.

(b) Prove that $f(x)=x^2-3x+2$ is injective in $(3/2,\infty)$.

(c) Prove that $f(x)=\dfrac{x}{x^2+1}$ is injective in $(1,\infty)$ and find $f^{-1}(\sqrt{2}/3)$.

(d) Determine if those same functions are surjective and bijective in their domains.

Problem 2.7 Calculate:

(i) $\arctan\frac{1}{2}+\arctan\frac{1}{3}$; (ii) $\arctan 2+\arctan 3$; (iii) $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}$.

HINT: Calculate the tangent of those expressions using the formula for the tangent of the sum and paying attention to the signs.

Problem 2.8 Simplify the following expressions:

(i) $f(x) = \sin(\arccos x)$;
(ii) $f(x) = \sin(2\arcsin x)$;
(iii) $f(x) = \tan(\arccos x)$;
(iv) $f(x) = \sin(2\arctan x)$;
(v) $f(x) = \cos(2\arctan x)$;
(vi) $f(x) = e^{4\log x}$.

Problem 2.9 Solve, for $x, y > 0$, the system of equations

$$\begin{cases} x^y = y^x, \\ y = 3x. \end{cases}$$

Problem 2.10

(a) Describe the function g in terms of f in the following cases ($c \in \mathbb{R}$ is a constant):

(i) $g(x) = f(x) + c$;
(ii) $g(x) = f(x+c)$;
(iii) $g(x) = f(cx)$;
(iv) $g(x) = f(1/x)$;
(v) $g(x) = f(|x|)$;
(vi) $g(x) = |f(x)|$;
(vii) $g(x) = 1/f(x)$;
(viii) $g(x) = [f(x)]_+ \equiv \max\{f(x), 0\}$.

(b) Plot the functions when $f(x) = x^2$.
(c) Plot the functions when $f(x) = \sin x$.

Problem 2.11 Sketch, using the fewest possible calculations, the graph of the following functions:

(i) $f(x) = (x+2)^2 - 1$;
(ii) $f(x) = \sqrt{4-x}$;
(iii) $f(x) = x^2 + \dfrac{1}{x}$;
(iv) $f(x) = \dfrac{1}{1+x^2}$;
(v) $f(x) = \min\{x, x^2\}$;
(vi) $f(x) = |e^x - 1|$;
(vii) $f(x) = \sqrt{|x| - x}$;
(viii) $f(x) = \dfrac{1}{\lfloor 1/x \rfloor}$;
(ix) $f(x) = |x^2 - 1|$;
(x) $f(x) = 1 - e^{-x}$;
(xi) $f(x) = \log(x^2 - 1)$;
(xii) $f(x) = x\sin(1/x)$.

HINT: In (viii) $\lfloor x \rfloor$ denotes the integer part of x, i.e., the largest integer $n \leqslant x$.

Problem 2.12

(a) Prove that $\cosh x$ is even and $\sinh x$ is odd.
(b) Prove the identities $\cosh^2 x - \sinh^2 x = 1$ and $\sinh(2x) = 2\sinh x \cosh x$.