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Calculus I

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Unit 3. Sequences

3. Sequences

3.1 Sequences of real numbers

A *sequence* is simply an infinite ordered list of real numbers

$$\{a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots\}.$$

(We normally use the convention of delimiting sequences by curly brackets.) Often the first terms of the sequence self-explain the rest of the sequence. This is the case of sequences such as

$$\{1,2,3,4,\ldots\}, \qquad \{1,1,1,1,\ldots\}, \qquad \{1,0,1,0,1,0,\ldots\}.$$

In most cases though, they are given by a formula as a function of n, e.g.,

$$a_n = n, \qquad b_n = (-1)^n, \qquad c_n = \frac{1}{n}, \qquad d_n = \left(1+\frac{1}{n}\right)^n.$$

Another possibility is to obtain a sequence through a recurrence. A recurrence is a formula that obtains the nth term in the sequence given the previous k terms (most often just one or two). For instance,

$$a_n = \sqrt{a_{n-1}+1}, \quad a_1 = 1.$$

Sometimes the solution of a recurrence is given by a formula, but most of the times it is not possible to find such a formula. Anyway, the recurrence provides a constructive way of describing the full sequence.

R We usually represent a sequence with the symbol $\{a_n\}_{n=1}^{\infty}$, where a_n is referred to as the *general term*, regardless of whether there is an explicit formula that provides a_n as a function of n or not.

In more rigorous terms we have the following definition of sequence:

Definition 3.1.1 A sequence is a map $f : \mathbb{N} \to \mathbb{R}$.

As a matter of fact, this is what the symbol a_n denotes: to each $n \in \mathbb{N}$ there correspond a real number a_n.

R Sequences can begin at $n = 0$ instead of $n = 1$, or in general at any index $n = k$, with $k = 0, 1, 2, \ldots$ Note that $\{a_n\}_{n=k}^{\infty} = \{a_{n+k-1}\}_{n=1}^{\infty}$, so that any sequence can be rewritten as a true map $f : \mathbb{N} \to \mathbb{R}$.

■ **Example 3.1 — Fibonacci sequence.** Leonardo di Pisa (c. 1170 – c. 1250), known as *Fibonacci* (son of Bonacci), was an Italian mathematician, considered to be "the most talented Western mathematician of the Middle Ages". Fibonacci popularized the Hindu-Arabic numeral system to the Western World mainly through his book, *Liber Abaci (Book of Calculation),* published in 1202.

An example of *Liber Abaci* is the well-known sequence of Fibonacci numbers. Fibonacci proposed the sequence as the solution to a problem of how a population of idealised rabbits grows generation after generation. His assumptions were:

1. Begin with one male-female couple of rabbits that have just been born.
2. Rabbits reach sexual maturity after one month.
3. The gestation period of a rabbit is one month.
4. After reaching sexual maturity, female rabbits give birth every month.
5. A female rabbit gives birth to one male rabbit and one female rabbit.
6. Rabbits do not die.

If F_n is the number of rabbit couples at month n, then $F_0 = 1$ (the initial couple has just been born), $F_1 = 1$ (rabbits need a month to reach sexual maturity) and then

$$F_{n+1} = F_n + F_{n-1}, \quad n > 0, \tag{3.1}$$

in other words, the number of couples next month (F_{n+1}) is the number of couples the current month (F_n) plus all new couples. There is a new couple for every sexually mature couple, and the sexually mature couples are those couples that existed last month (F_{n-1}), because it takes a month to reach sexual maturity.

Iterating equation (3.1) we get the *Fibonacci sequence*

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}.$$

■

Definition 3.1.2 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that this sequence
(a) **increases monotonically** if $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$;
(b) **decreases monotonically** if $a_n \geqslant a_{n+1}$ for all $n \in \mathbb{N}$;
(c) is **alternating** if $(a_{n+1} - a_n)(a_n - a_{n-1}) < 0$ for all $n \in \mathbb{N}$ (i.e., it goes up and down alternatively);
(d) is **bounded from above** if there exists $c \in \mathbb{R}$ such that $a_n \leqslant c$ for all $n \in \mathbb{N}$;
(e) is **bounded from below** if there exists $c \in \mathbb{R}$ such that $a_n \geqslant c$ for all $n \in \mathbb{N}$.

■ **Example 3.2** Let us consider the sequence $a_n = \dfrac{n}{n+1}$. It is easy to see that it is bounded from above by 1, because the denominator is always larger than the numerator. Another way to see it is by looking for a c such that $a_n < c$, i.e.,

$$\frac{n}{n+1} < c \quad \Leftrightarrow \quad n < cn + c.$$

Clearly this second inequality holds if we take $c = 1$.

Let us now prove that $\{a_n\}_{n=1}^{\infty}$ increases monotonically. To check for monotonicity it is normally easier to check for the sign of $a_n - a_{n+1}$ or whether a_n/a_{n+1} is larger or smaller than 1. Let us try the former:

$$a_n - a_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2} = \frac{n(n+2)-(n+1)^2}{(n+1)(n+2)} = \frac{n^2+2n-n^2-2n-1}{(n+1)(n+2)} = \frac{-1}{(n+1)(n+2)} < 0,$$

therefore $a_n < a_{n+1}$ and so the sequence increases monotonically. ■

Definition 3.1.3 — Subsequence. A **subsequence** of a sequence $\{a_n\}_{n=1}^{\infty}$ is any choice $\{a_{n_k}\}_{k=1}^{\infty}$ of infinitely many elements of the sequence. (n_k is the rule that tells what is the kth element of the subsequence.)

■ **Example 3.3** The sequence $\left\{\frac{1}{2k-1}\right\}_{k=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. The rule $n_k = 2k-1$ tells that we are selecting only the odd terms of the sequence. ■

■ **Example 3.4** The sequence $\left\{\frac{1}{2^{k-1}}\right\}_{k=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. The rule $n_k = 2^{k-1}$ tells that we are selecting only the terms of the sequence whose index is a power of 2. ■

Proposition 3.1.1 Every sequence has at least one monotonic subsequence (either increasing or decreasing).

3.2 Limit of a sequence

Consider the sequence $a_n = \dfrac{n}{n+1}$. Let us explicitly display some of its terms:

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots, \frac{1000}{1001}, \ldots\right\} = \{0.5000,\ 0.6667,\ 0.7500,\ 0.8000, \ldots,\ 0.9990, \ldots\}$$

where we have calculated the numbers to an accuracy of four decimal places. These numbers as well as Figure 3.2 both illustrate the fact that the more we increase n the closer is a_n to the value 1. Sequences exhibiting this behavior are said to have a limit. In this case, the limit of a_n is 1.

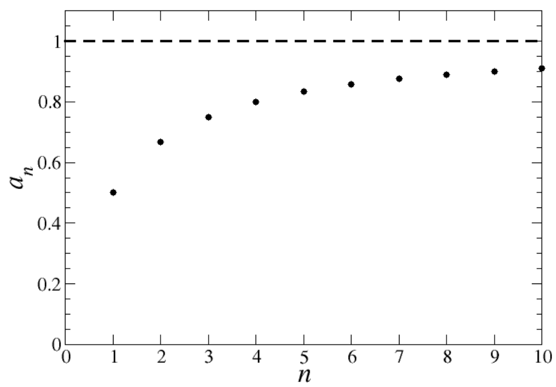

Figure 3.1: Plot of the first ten terms of the sequence $a_n = \dfrac{n}{n+1}$.

We need a more precise definition of limit that captures this idea in all its flavour. To this purpose we have the following definition:

Definition 3.2.1 — Limit of a sequence. The real number a is said to be the **limit** of the sequence $\{a_n\}$ if for any real number $\varepsilon > 0$ —no matter how small— there exits an index N —which may depend on ε— such that for every $n > N$ the sequence satisfies the inequality

$$|a_n - a| < \varepsilon. \tag{3.2}$$

The sequence is then said to be **convergent.**

■ **Example 3.5** Let us apply the definition to actually prove that the limit of $a_n = \dfrac{n}{n+1}$ is 1. Let $\varepsilon > 0$ be a given arbitrary number. We need to find for which indices n the inequality

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon \tag{3.3}$$

holds. Clearly,

$$\frac{n}{n+1} - 1 = \frac{n-(n+1)}{n+1} = \frac{-1}{n+1} \quad \Rightarrow \quad \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1}.$$

Thus inequality (3.3) is equivalent to

$$\frac{1}{n+1} < \varepsilon \quad \Leftrightarrow \quad n+1 > \frac{1}{\varepsilon} \quad \Leftrightarrow \quad n > \frac{1}{\varepsilon} - 1.$$

We have the proof we wanted. Suppose $\varepsilon = 0.1$. Then

$$\frac{1}{\varepsilon} - 1 = \frac{1}{0.1} - 1 = 10 - 1 = 9,$$

so we can take $N = 9$ and the definition applies. Suppose $\varepsilon = 0.01$. Then

$$\frac{1}{\varepsilon} - 1 = \frac{1}{0.01} - 1 = 100 - 1 = 99,$$

so we can take $N = 99$ and again the definition applies.

It is clear that we can take ε smaller and smaller, and that will imply that N is larger and larger, but nevertheless, no matter how small ε is taken, there always exists N satisfying the definition. ■

We can also characterise sequences like $a_n = n$, which not only do not have a limit, but they grow without bound as n increases.

Definition 3.2.2 — Divergent sequence. The sequence $\{a_n\}$ is said to be **divergent to** $+\infty$ if for any real number $C > 0$ —no matter how large— there exits an index N —which may depend on C— such that for every $n > N$ the sequence satisfies the inequality

$$a_n > C. \tag{3.4}$$

Likewise, it is said to be **divergent to** $-\infty$ if for any real number $C < 0$ there exits an index N such that for every $n > N$ the sequence satisfies the inequality

$$a_n < C. \tag{3.5}$$

R We denote the limit of a sequence with the symbol $\lim_{n\to\infty} a_n = a$ if it converges, or $\lim_{n\to\infty} a_n = \pm\infty$ if it diverges to $+\infty$ or $-\infty$.

■ **Example 3.6** The sequence $a_n = n^p$ diverges to $+\infty$ if $p > 0$, and converges if $p \leqslant 0$. We will prove it by applying the definition.

Let $p > 0$ and $C > 0$ and consider the inequality

$$a_n = n^p > C \qquad \Leftrightarrow \qquad n > C^{1/p}.$$

In other words, if $N > C^{1/p}$, then for every $n > N$ we will have $a_n > C$.

Let now $p = 0$. Then $a_n = 1$ for all $n \in \mathbb{N}$, and therefore

$$\lim_{n\to\infty} a_n = 1.$$

Finally, let $p = -q < 0$ and $\varepsilon > 0$. Then,

$$|a_n - 0| = \left|\frac{1}{n^q} - 0\right| = \frac{1}{n^q} < \varepsilon \qquad \Leftrightarrow \qquad n^q > \frac{1}{\varepsilon} \qquad \Leftrightarrow \qquad n > \left(\frac{1}{\varepsilon}\right)^{1/q}.$$

So if we take $N > (1/\varepsilon)^{1/q}$ then for every $n > N$ we will have $|a_n - 0| < \varepsilon$. Therefore

$$\lim_{n\to\infty} a_n = 0.$$

■

Aside from convergent and divergent sequences there are sequences that neither converge nor diverge. This are simply said to be **nonconvergent** sequences.

■ **Example 3.7** The sequence $a_n = (-1)^n n$ is nonconvergent, because $|a_n| = n$ is divergent, but a_n alternates sign. ■

Proposition 3.2.1 If the limit of $\{a_n\}_{n=1}^{\infty}$ exists, it is unique.

Proof. Suppose that a and b ($a < b$) are two limits of the same sequence. Then, according to the definition, for every $\varepsilon > 0$

$$|a_n - a| < \varepsilon, \qquad |a_n - b| < \varepsilon.$$

In other words,

$$a_n < a + \varepsilon, \qquad b - \varepsilon < a_n.$$

But if ε is such that $a + \varepsilon < b - \varepsilon$ this two inequalities cannot hold at the same time. And this can be accomplished by any $\varepsilon < (b - a)/2$. Hence b and a cannot be two different numbers. ■

Proposition 3.2.2 Every subsequence of a convergent sequence has the same limit as the sequence.

Applying the definition to prove that a sequence has a limit may be a daunting task. For that reason we normally apply some properties that all convergent sequences satisfy in order to simplify the problem. These are some algebraic properties of the limits:

Proposition 3.2.3 Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two convergent sequences with limits a and b respectively. Then the following properties hold:

1. $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$;
2. $\lim_{n\to\infty} a_n b_n = ab$;
3. if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$;
4. $\lim_{n\to\infty} a_n^{b_n} = a^b$;
5. $\lim_{n\to\infty} \log a_n = \log a$.

Two further theorems turn out to be very practical for calculating limits.

Theorem 3.2.4 — Sandwich rule. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$, and for all $n > N$, for some $N \in \mathbb{N}$, we have $a_n \leqslant b_n \leqslant c_n$, then also $\lim_{n\to\infty} b_n = l$.

In particular this implies that if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$ too because $-|a_n| \leqslant a_n \leqslant |a_n|$.

Theorem 3.2.5 Every monotonically increasing (respectively decreasing) sequence bounded from above (respectively below) converges to some limit.

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ increases monotonically and is bounded above. The *supremum property* (Theorem 1.3.2(i)) guarantees that $\{a_n\}_{n=1}^{\infty}$ has a supremum a. So $a_n \leqslant a$ for all $n \in \mathbb{N}$ and no other real number $c < a$ is an upper bound. Let us take $\varepsilon > 0$ and set $c = a - \varepsilon$. This is not an upper bound, therefore there must be some $N \in \mathbb{N}$ such that $a_N > a - \varepsilon$. But since the sequence is increasing we then have

$$a - \varepsilon < a_N \leqslant a_{N+1} \leqslant a_{N+2} \leqslant \cdots \leqslant a < a + \varepsilon.$$

In other words, $|a_n - a| < \varepsilon$ for all $n > N$. ■

As the proof shows, this theorem is a consequence of the completeness of real numbers. Intuitively it makes perfect sense, for if a sequence keeps on increasing but cannot trespass a certain bound, it must converge to something. And as a matter of fact, the proof shows that this "something" is precisely the supremum of the sequence, if it increases, or its infimum, if it decreases.

■ **Example 3.8** Consider the sequence

$$a_n = \frac{\sin n}{n}.$$

In principle a_n is the quotient of two sequences, namely $b_n = \sin n$ and $c_n = n$. However, none of them has a limit: c_n diverges to $+\infty$ and b_n exhibits a random behavior, as illustrated by Figure 3.3.

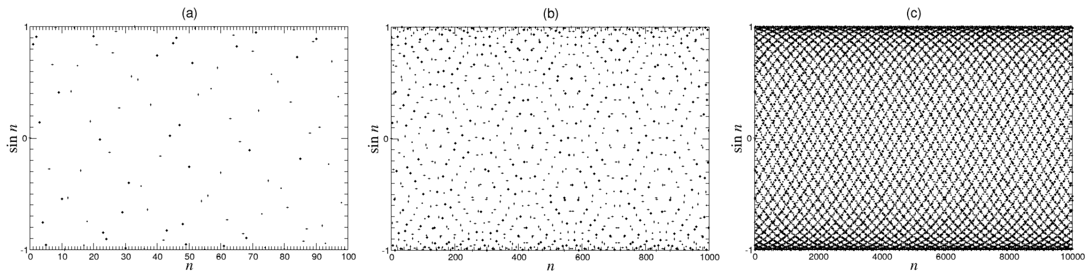

Figure 3.2: Plot of $\sin n$ at three different scales: (a) for $1 \leqslant n \leqslant 100$, (b) for $1 \leqslant n \leqslant 1000$, and (c) for $1 \leqslant n \leqslant 10000$.

However, it is always true that, irrespective of n, $-1 \leqslant \sin n \leqslant 1$, so

$$-\frac{1}{n} \leqslant a_n \leqslant \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, applying the sandwich rule we conclude that $\lim_{n\to\infty} a_n = 0$. ■

■ **Example 3.9** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence

$$a_{n+1} = \frac{1}{2}(a_n + 6), \qquad a_1 = 2.$$

The first few terms look like

$$\left\{2,4,5,\frac{11}{2},\frac{23}{4},\frac{47}{8},\frac{95}{16},\dots\right\} = \{2,\ 4,\ 5,\ 5.5,\ 5.75,\ 5.875,\ 5.9375,\dots\},$$

so it seems that it does converge toward 6 and is monotonically increasing.

Let us first show, by induction, that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 6. To begin with, $a_1 = 2 < 6$. Now suppose $a_n < 6$. Then

$$a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6+6) = 6,$$

so if $a_n < 6$ then also $a_{n+1} < 6$.

Finally let us show that the sequence is monotonically increasing. For that, let us calculate

$$a_{n+1} - a_n = \frac{1}{2}(a_n + 6) - a_n = \frac{1}{2}(6 - a_n) > 0$$

because we have shown that $a_n < 6$. Therefore $a_{n+1} > a_n$, and because of the theorem, we have proven that the sequence converges to some number a. To actually determine a we must take the limit in the recurrence as

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right) \quad \Leftrightarrow \quad a = \frac{1}{2}(a+6) \quad \Leftrightarrow \quad 2a = a + 6 \quad \Leftrightarrow \quad a = 6.$$

■

■ **Example 3.10** Let us consider the sequence

$$a_n = \frac{3n^2 + 2n - 1}{5n^4 - 2n + 7}.$$

Calculating its limit through the definition is very difficult. However we can play the following trick: we will factor out the largest power both in the numerator and in the denominator,

$$a_n = \frac{n^2\left(3 + \frac{2}{n} - \frac{1}{n^2}\right)}{n^4\left(5 - \frac{2}{n^3} + \frac{7}{n^4}\right)} = \frac{n^2}{n^4} \cdot \frac{3 + \frac{2}{n} - \frac{1}{n^2}}{5 - \frac{2}{n^3} + \frac{7}{n^4}} = \frac{1}{n^2} \cdot \frac{3 + \frac{2}{n} - \frac{1}{n^2}}{5 - \frac{2}{n^3} + \frac{7}{n^4}},$$

and now we can apply the algebraic rules for the limits. Since for any $p > 0$

$$\lim_{n\to\infty} \frac{1}{n^p} = 0,$$

then

$$\lim_{n\to\infty}\left(3 + \frac{2}{n} - \frac{1}{n^2}\right) = 3, \qquad \lim_{n\to\infty}\left(5 - \frac{2}{n^3} + \frac{7}{n^4}\right) = 5,$$

and therefore

$$\lim_{n\to\infty} a_n = 0 \cdot \frac{3}{5} = 0.$$

■

From the example above we can infer the following proposition:

Proposition 3.2.6 Let

$$a_n = \frac{\alpha_p n^p + \alpha_{p-1} n^{p-1} + \cdots + \alpha_1 n + \alpha_0}{\beta_q n^q + \beta_{q-1} n^{q-1} + \cdots + \beta_1 n + \beta_0}.$$

Then
(a) if $q = p$, $\lim_{n\to\infty} a_n = \frac{\alpha_p}{\beta_q}$;
(b) if $q > p$, $\lim_{n\to\infty} a_n = 0$;
(c) if $q < p$, $\lim_{n\to\infty} a_n = +\infty$ if $\alpha_p\beta_q > 0$ and $\lim_{n\to\infty} a_n = -\infty$ if $\alpha_p\beta_q < 0$.

Proof. (a) and (b) are proven as in the example. For (c) we can prove equivalently that $\lim_{n\to\infty} \frac{1}{a_n} = 0$ because it reduces to (b). The sign is the sign of the quotient α_p/β_q —which is the same as that of the product $\alpha_p\beta_q$. ■

Here is a list of some important limits that often occur in calculations:

1. $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for all $a > 0$.
2. $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ for all $p \in \mathbb{R}$.
3. $\lim_{n\to\infty} a^n = 0$ for all $|a| < 1$ and $\lim_{n\to\infty} a^n = +\infty$ for all $a > 1$.
4. $\lim_{n\to\infty} \frac{n^r}{a^n} = 0$ for all $|a| > 1$ and $r \in \mathbb{R}$.

Finally, there is a very important result that at this point will be very easy to prove:

Theorem 3.2.7 — Bolzano-Weierstrass theorem. Every bounded sequence has at least one convergent subsequence.

Proof. The proof is very simple. From Proposition 3.1.1 we know that every sequence has at least one monotonic subsequence. This subsequence will be bounded because the whole sequence is bounded. Therefore this subsequence must converge to some limit by virtue of Theorem 3.2.5. ■

R Note that this result holds even for sequences without limit.

■ **Example 3.11** The sequence $\{(-1)^n\}_{n=1}^{\infty}$ does not converge; however it is bounded, and the subsequence $\{1\}_{n=1}^{\infty}$ containing only the even terms clearly converges to 1. ■

3.3 Number *e*

There is a special convergent sequence whose limit defines an irrational number of great importance in mathematics. It is the following:

Definition 3.3.1

$$e \equiv \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845904523536028747135266249775 72\ldots \tag{3.6}$$

The sequence is monotonically increasing and bounded above, but converges very slowly to its limit, as Table 3.1 shows. Both the prove of the convergence of this sequence and that of the irrationality of its limit appear in Apppendix C.

n	$\left(1+\frac{1}{n}\right)^n$
1	**2**
2	**2**.25
5	**2**.48832
10	**2**.59374246
100	**2.7**04813829
1000	**2.71**6923932
10000	**2.718**145927
100000	**2.7182**68237

Table 3.1: Some values of the sequence $\left(1+\frac{1}{n}\right)^n$. Note that n must increase by an order of magnitude in order to obtain a new decimal figure of e.

Number e is not only an irrational number, but a transcendental one. This means that there is no algebraic equation whose solution is e (in particular, e cannot be expressed in terms of radicals).

Many limits involve e. Here are a few examples:

■ **Example 3.12** Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}.$$

Since we can factor out

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right),$$

then

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\right) = e.$$

■

■ **Example 3.13** Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}.$$

The sequence

$$\left(1+\frac{1}{k^2+1}\right)^{k^2+1}, \quad k\in\mathbb{N},$$

is a subsequence of

$$\left(1+\frac{1}{n}\right)^{n}$$

(that corresponding to those n such that $n=k^2+1$). Any subsequence of a convergent sequence has the same limit as the original sequence, therefore

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1} = e.$$

■

■ **Example 3.14** Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3}.$$

We can rewrite

$$\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3}=\left[\left(1+\frac{1}{3n^2+1}\right)^{3n^2+1}\right]^{\frac{2n^2-3}{3n^2+1}},$$

so that

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3}=\left[\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{3n^2+1}\right]^{\lim_{n\to\infty}\frac{2n^2-3}{3n^2+1}}.$$

Now, since

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{3n^2+1}=e,\qquad \lim_{n\to\infty}\frac{2n^2-3}{3n^2+1}=\frac{2}{3},$$

we finally have

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3}=e^{2/3}.$$

■

It is easy to generalise this example and show that any limit of the form

$$\lim_{n\to\infty}(1+b_n)^{c_n},$$

where

$$\lim_{n\to\infty}b_n=0,\qquad \lim_{n\to\infty}c_n=\infty,$$

can be calculated as

$$\lim_{n\to\infty}(1+b_n)^{c_n}=e^{\left(\lim\limits_{n\to\infty}b_nc_n\right)}.$$

■ **Example 3.15** Calculate

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}.$$

We can rewrite

$$\left(1-\frac{1}{n}\right)^{n}=\left(\frac{n-1}{n}\right)^{n}=\left(\frac{n-1}{1+(n-1)}\right)^{n}=\frac{1}{\left(\frac{1+(n-1)}{n-1}\right)^{n}}=\frac{1}{\left(1+\frac{1}{n-1}\right)^{n}}.$$

Therefore

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}=\frac{1}{e^{\left(\lim\limits_{n\to\infty}\frac{n}{n-1}\right)}}=\frac{1}{e}=e^{-1}.$$

Incidentally, this example shows that the above argument is valid regardless of the sign of b_n. ■

As a matter of fact, number e is involved in any limit corresponding to what is referred to as an *indeterminacy* of type 1^∞. This is a short way to refer to the limit of a sequence of the form $a_n^{c_n}$, where $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} c_n = \infty$. Here is an example:

■ **Example 3.16** Calculate

$$\lim_{n\to\infty}\left(\frac{3n^2+1}{(3n+2)(n-3)}\right)^{2n+1}.$$

First we check that

$$\lim_{n\to\infty}\frac{3n^2+1}{(3n+2)(n-3)} = \lim_{n\to\infty}\frac{3n^2+1}{3n^2-7n-6} = 1, \qquad \lim_{n\to\infty}(2n+1) = \infty,$$

so we are dealing with a 1^∞ indeterminacy. The way to proceed is always the same. We rewrite

$$\frac{3n^2+1}{(3n+2)(n-3)} = \frac{3n^2+1}{3n^2-7n-6} = 1+\left(\frac{3n^2+1}{3n^2-7n-6}-1\right) = 1+\frac{7n+7}{3n^2-7n-6},$$

so that we can transform the original limit into

$$\lim_{n\to\infty}\left(\frac{3n^2+1}{(3n+2)(n-3)}\right)^{2n+1} = \lim_{n\to\infty}\left(1+\frac{7n+7}{3n^2-7n-6}\right)^{2n+1}.$$

Now

$$\lim_{n\to\infty}\frac{7n+7}{3n^2-7n-6} = 0,$$

therefore

$$\lim_{n\to\infty}\left(\frac{3n^2+1}{(3n+2)(n-3)}\right)^{2n+1} = e^c,$$

where

$$c = \lim_{n\to\infty}\frac{(7n+7)(2n+1)}{3n^2-7n-6} = \lim_{n\to\infty}\frac{14n^2+21n+7}{3n^2-7n-6} = \frac{14}{3}.$$

■

3.4 Indeterminacies

Apart from the 1^∞ indeterminacy we have just encountered and which is related to the number e, there are other indeterminacies that often appear when we calculate limits. They are basically these:

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad 0\cdot\infty, \qquad 0^0, \qquad \infty^0, \qquad \infty-\infty.$$

Their meaning is similar to that of the 1^∞ indeterminacy. For instance, $0\cdot\infty$ just denotes the case where we find a limit such as $\lim_{n\to\infty} a_n b_n$, where $\lim_{n\to\infty} a_n = 0$, and $\lim_{n\to\infty} b_n = \infty$ (and similarly for the other cases).

R Note that the following expressions are not indeterminacies, but well defined limits:

$$\infty^\infty = \infty, \qquad 0^\infty = 0, \qquad \infty+\infty = \infty, \qquad \frac{0}{\infty} = 0, \qquad \frac{\infty}{0} = \pm\infty.$$

Indeterminacies are related to one another. For instance, if $\lim_{n\to\infty} a_n = \infty$, and $\lim_{n\to\infty} b_n = \infty$, then

$$\underbrace{\lim_{n\to\infty} \frac{a_n}{b_n}}_{\frac{\infty}{\infty}} = \underbrace{\lim_{n\to\infty} a_n b_n^{-1}}_{\infty\cdot 0} = \underbrace{\lim_{n\to\infty} \frac{b_n^{-1}}{a_n^{-1}}}_{\frac{0}{0}}.$$

Indeterminacies must be solved case by case —there is no general rule to apply. We have already found the $\frac{\infty}{\infty}$ indeterminacy in the case of rational sequences, and as Proposition 3.2.6 shows, the solution depens on the specific sequence whose limit we want to calculate.

Other cases of this indeterminacy as well as the $\frac{0}{0}$ indeterminacy require the following result:

Theorem 3.4.1 — Stolz theorem. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. Suppose that $\{b_n\}_{n=1}^{\infty}$ is strictly monotonic (increasing or decreasing) and either:

(a) $\lim_{n\to\infty} b_n = \pm\infty$, or

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

Then

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \ell \quad \Rightarrow \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \ell. \tag{3.7}$$

From this theorem we can derive three important corollaries:

Corollary 3.4.2

$$\lim_{n\to\infty} c_n = c \quad \Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}(c_1 + c_2 + \cdots + c_n) = c.$$

Proof. Take the sequences $a_n = c_1 + c_2 + \cdots + c_n$ and $b_n = n$, and note that b_n is strictly increasing and diverges. Then

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{(c_1 + c_2 + \cdots + c_{n-1} + c_n) - (c_1 + c_2 + \cdots + c_{n-1})}{n - (n-1)} = \lim_{n\to\infty} c_n = c,$$

therefore

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c.$$

■

Corollary 3.4.3

$$\lim_{n\to\infty} c_n = c \quad \Rightarrow \quad \lim_{n\to\infty} \sqrt[n]{c_1 c_2 \cdots c_n} = c.$$

Proof. Taking logarithms

$$\log \sqrt[n]{c_1 c_2 \cdots c_n} = \frac{1}{n}(\log c_1 + \log c_2 + \cdots + \log c_n).$$

Since $\lim_{n\to\infty} \log c_n = \log c$, applying the previous corollary we obtain

$$\lim_{n\to\infty} \log \sqrt[n]{c_1 c_2 \cdots c_n} = \lim_{n\to\infty} \frac{1}{n}(\log c_1 + \log c_2 + \cdots + \log c_n) = \log c,$$

and therefore

$$\lim_{n\to\infty} \sqrt[n]{c_1 c_2 \cdots c_n} = c.$$

■

Corollary 3.4.4

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = a \qquad \Rightarrow \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = a.$$

Proof. We can write

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} a_1,$$

so if we define $c_n \equiv a_n / a_{n-1}$ for $n > 1$ and $c_1 = a_1$ we have

$$a_n = c_1 c_2 \cdots c_n.$$

But $\lim_{n\to\infty} c_n = a$, therefore the previous corollary implies

$$\lim_{n\to\infty} \sqrt[n]{c_1 c_2 \cdots c_n} = \lim_{n\to\infty} \sqrt[n]{a_n} = a.$$

■

■ **Example 3.17** We can easily show that $\lim_{n\to\infty} \sqrt[n]{a} = 1$, for all $a > 0$, as an application of the last corollary, because

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{a}{a} = 1.$$

Similarly we can prove that $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$, for all $p \in \mathbb{R}$, because

$$\lim_{n\to\infty} \sqrt[n]{n^p} = \lim_{n\to\infty} \frac{n^p}{(n-1)^p} = \left(\lim_{n\to\infty} \frac{n}{n-1} \right)^p = 1^p = 1.$$

■

■ **Example 3.18** We can calculate[1]

$$\lim_{n\to\infty} \sqrt[n]{\binom{2n}{n}}$$

using the third corollary as

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{\binom{2n-2}{n-1}} = \lim_{n\to\infty} \frac{(2n)!}{n!n!} \frac{(n-1)!(n-1)!}{(2n-2)!} = \lim_{n\to\infty} \frac{2n(2n-1)}{n^2} = \lim_{n\to\infty} \frac{4n-2}{n} = 4,$$

where we have used the fact that $(2n)! = 2n(2n-1)\cdot(2n-2)!$ and $n! = n\cdot(n-1)!$. ■

Stolz theorem is particularly useful when one of the sequences involved is a sum of n terms, as in this example:

[1] The symbols $\binom{n}{k}$ represent combinatorial coefficients. Their definition and properties can be found in Appendix B.

■ **Example 3.19** Calculate

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\log n}.$$

An application of the theorem transforms this limit into

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\log n - \log(n-1)} = \lim_{n\to\infty} \frac{1}{n\log\left(\frac{n}{n-1}\right)} = \lim_{n\to\infty} \frac{1}{\log\left(\frac{(n-1)+1}{n-1}\right)^n} = \lim_{n\to\infty} \frac{1}{\log\left(1+\frac{1}{n-1}\right)^n}.$$

But we know that

$$\lim_{n\to\infty} \left(1+\frac{1}{n-1}\right)^n = e,$$

and that $\log e = 1$, therefore

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\log n} = 1.$$

■

The indeterminacy $\infty - \infty$ is a particularly difficult one, but often can be solved by an algebraic manipulation of the expressions.

■ **Example 3.20** Calculate

$$\lim_{n\to\infty} \left(\sqrt{n^2+2n-1} - \sqrt{n^2-3}\right).$$

Both terms in this expression are divergent sequences, so this is an $\infty - \infty$ indeterminacy. The trick to calculate this limit is to make use of the identity

$$x^2 - y^2 = (x-y)(x+y) \qquad \Rightarrow \qquad x - y = \frac{x^2-y^2}{x+y}.$$

In this case

$$\sqrt{n^2+2n-1} - \sqrt{n^2-3} = \frac{n^2+2n-1-(n^2-3)}{\sqrt{n^2+2n-1}+\sqrt{n^2-3}} = \frac{2n+2}{n\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}+n\sqrt{1-\frac{3}{n^2}}}$$

$$= \frac{2+\frac{2}{n}}{\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}+\sqrt{1-\frac{3}{n^2}}} \xrightarrow[n\to\infty]{} \frac{2}{1+1} = 1.$$

In the case that there are higher order roots involved, in may be useful to apply the generalised identity

$$x^{p+1} - y^{p+1} = (x-y)(x^p + x^{p-1}y + x^{p-2}y^2 + \cdots + xy^{p-1} + y^p)$$

that leads to

$$x - y = \frac{x^{p+1}-y^{p+1}}{x^p + x^{p-1}y + x^{p-2}y^2 + \cdots + xy^{p-1} + y^p}.$$

■

3.5 Asymptotic comparison of sequences

In order to solve indeterminacies it is handy to compare sequences for large values of n. If they behave similarly they can often be replaced by each other. If one is negligible with respect to the other, it is the second one that decides the trend. Let us formalise this notion:

Definition 3.5.1 — Asymptotic comparison. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ two sequences which either both diverge or both converge to 0. We say that

(a) a_n and b_n are **equivalent** (we denote it $a_n \sim b_n$) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1;$$

(b) a_n is **negligible** compared to b_n (we denote it $a_n \ll b_n$) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

■ **Example 3.21** Example 3.19 proves that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \log n, \tag{3.8}$$

(which incidentally proves that

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \infty, \tag{3.9}$$

a fact that will be of utmost relevance in the next chapter). ■

■ **Example 3.22** Suppose that $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence that converges to 0. We have seen that

$$\lim_{n\to\infty} (1+\varepsilon_n)^{1/\varepsilon_n} = e.$$

Taking logarithms

$$\lim_{n\to\infty} \frac{\log(1+\varepsilon_n)}{\varepsilon_n} = 1;$$

in other words, $\log(1+\varepsilon_n) \sim \varepsilon_n$.

We can now define the new sequence $\delta_n \equiv \log(1+\varepsilon_n)$, whose limit is 0. Then $\varepsilon_n = e^{\delta_n} - 1$, so we transform the previous limit into

$$1 = \lim_{n\to\infty} \frac{\log(1+\varepsilon_n)}{\varepsilon_n} = \lim_{n\to\infty} \frac{\delta_n}{e^{\delta_n} - 1}.$$

Therefore $e^{\delta_n} - 1 \sim \delta_n$. ■

We can infer further equivalences from geometric arguments. Figure 3.4 shows that[2] $|\sin x| \leqslant |x| \leqslant |\tan x|$ for all $-\pi/2 \leqslant x \leqslant \pi/2$ (we take absolute values because all quantities become negative for negative x, but the relation between the lengths remains true). From the first inequality we conclude that, within this interval,

$$\left|\frac{\sin x}{x}\right| \leqslant 1.$$

[2] The first inequality, $|\sin x| \leqslant |x|$, is obvious from the figure. The second one, $|x| \leqslant |\tan x|$ follows because the area of the sector is obviously smaller than that of the big triangle. Now, the former is $\frac{1}{2}|x|$ whereas the latter is $\frac{1}{2}|\tan x|$, hence the inequality.

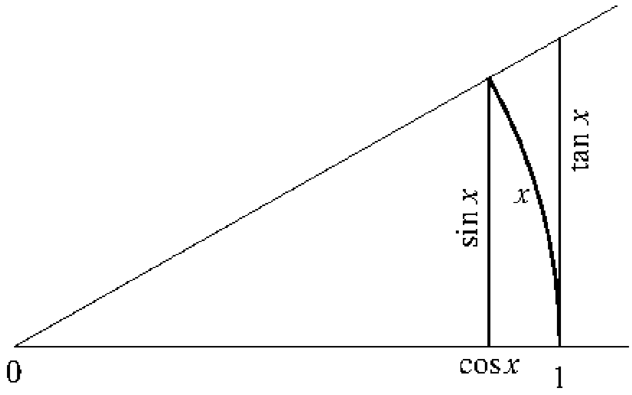

Figure 3.3: Comparison of $\sin x$, x, and $\tan x$ for small $0 \leqslant x \leqslant \pi/2$.

But $\dfrac{\sin x}{x} \geqslant 0$ for all $-\pi/2 \leqslant x \leqslant \pi/2$ (because both $\sin x$ and x have the same sign), so

$$0 \leqslant \frac{\sin x}{x} \leqslant 1.$$

From the second inequality,

$$|x| \leqslant \frac{|\sin x|}{|\cos x|} \qquad \Leftrightarrow \qquad |\cos x| \leqslant \left|\frac{\sin x}{x}\right|.$$

But since $\dfrac{\sin x}{x} \geqslant 0$ and $\cos x \geqslant 0$ for all $-\pi/2 \leqslant x \leqslant \pi/2$,

$$\cos x \leqslant \frac{\sin x}{x} \leqslant 1.$$

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence that converges to 0. From the inequality $|\sin \varepsilon_n| \leqslant |\varepsilon_n|$ (which is equivalent to $-|\varepsilon_n| \leqslant \sin \varepsilon_n \leqslant |\varepsilon_n|$) and the sandwich rule we conclude that

$$\lim_{n\to\infty} \sin \varepsilon_n = 0. \tag{3.10}$$

On the other hand, $\cos \varepsilon_n = \sqrt{1-\sin^2 \varepsilon_n}$, so

$$\lim_{n\to\infty} \cos \varepsilon_n = 1. \tag{3.11}$$

Since we have

$$\cos \varepsilon_n \leqslant \frac{\sin \varepsilon_n}{\varepsilon_n} \leqslant 1,$$

again using the sandwich rule we conclude that

$$\lim_{n\to\infty} \frac{\sin \varepsilon_n}{\varepsilon_n} = 1, \tag{3.12}$$

or equivalently that $\sin \varepsilon_n \sim \varepsilon_n$.

Also,

$$\lim_{n\to\infty} \frac{\tan \varepsilon_n}{\varepsilon_n} = \lim_{n\to\infty} \frac{\sin \varepsilon_n}{\varepsilon_n \cos \varepsilon_n} = 1,$$

therefore $\tan \varepsilon_n \sim \varepsilon_n$.

In summary, all these sequences are equivalent:

$$\log(1+\varepsilon_n) \sim (e^{\varepsilon_n}-1) \sim \sin\varepsilon_n \sim \tan\varepsilon_n \sim \varepsilon_n. \tag{3.13}$$

Exercise 3.1 Using the identities

$$\cos^2 x + \sin^2 x = 1, \qquad \cos^2 x - \sin^2 x = \cos 2x,$$

prove that

$$1-\cos\varepsilon_n \sim \frac{\varepsilon_n^2}{2}. \tag{3.14}$$

■

A very important equivalence is given in the following theorem:

Theorem 3.5.1 — Stirling formula.

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}. \tag{3.15}$$

The correct use of equivalences is as follows. Suppose $a_n \sim c_n$ and $b_n \sim d_n$. Then

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{a_n}{c_n}\frac{b_n}{d_n} c_n d_n = \underbrace{\lim_{n\to\infty} \frac{a_n}{c_n}}_{=1} \underbrace{\lim_{n\to\infty} \frac{b_n}{d_n}}_{=1} \lim_{n\to\infty} c_n d_n = \lim_{n\to\infty} c_n d_n.$$

So sequences can be replaced by equivalent sequences in products (and it can easily be shown that also in quotients).

Exercise 3.2 Calculate

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\log\left(\frac{n+1}{n}\right)}.$$

■

However, using equivalences in differences can lead to incorrect results. This example is illustrative of the sort of problems one can meet.

■ **Example 3.23** We want to calculate

$$\lim_{n\to\infty}\left(\sqrt{n^4+n^2}-n^2-1\right).$$

Proceeding as in Example 3.20,

$$\sqrt{n^4+n^2}-n^2-1 = \frac{n^4+n^2-(n^2+1)^2}{\sqrt{n^4+n^2}+n^2+1} = \frac{n^4+n^2-n^4-2n^2-1}{n^2\sqrt{1+\frac{1}{n^2}}+n^2\left(1+\frac{1}{n^2}\right)} = \frac{-1-\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^2}}+1+\frac{1}{n^2}}$$

therefore

$$\lim_{n\to\infty}\left(\sqrt{n^4+n^2}-n^2-1\right) = -\frac{1}{2}.$$

However, $\sqrt{n^4+n^2} \sim n^2$ because

$$\frac{\sqrt{n^4+n^2}}{n^2} = \sqrt{1+\frac{1}{n^2}} \xrightarrow[n\to\infty]{} 1,$$

so we might be tempted to reason as follows:

$$\sqrt{n^4+n^2}-n^2-1 \sim n^2-n^2-1=-1,$$

in which case we would conclude *incorrectly* that

$$\lim_{n\to\infty}\left(\sqrt{n^4+n^2}-n^2-1\right)=-1.$$

■

The problem that this example illustrates is that in replacing a sequence by an equivalent one we are ignoring smaller terms, which may become relevant if the dominant terms cancel out —as it usually happens in $\infty-\infty$ indeterminacies.

Negligible sequences, on the contrary, are relevant in sums and differences. For suppose $b_n \ll a_n$ and we want to calculate

$$\lim_{n\to\infty}\frac{c_n}{a_n+b_n}=\lim_{n\to\infty}\frac{c_n}{a_n\left(1+\frac{b_n}{a_n}\right)}=\lim_{n\to\infty}\frac{c_n}{a_n},$$

because

$$\lim_{n\to\infty}\frac{b_n}{a_n}=0.$$

In other words, in an expression like a_n+b_n we can simply eliminate the negligible sequence.

■ **Example 3.24** Let us show that

$$\log(3n^6-5n^2+2)\sim 6\log n.$$

Note that $2 \ll 5n^2 \ll 3n^6$, therefore

$$\log(3n^6-5n^2+2)\sim \log(3n^6)=\log 3+6\log n.$$

But $\log 3 \ll 6\log n$, hence the equivalence. ■

There is a hierarchy of negligible sequences which turns out to be very useful in calculations:

For any $a,b>0$, and $c>1$, it holds

$$(\log n)^a \ll n^b \ll c^n \ll n! \ll n^n. \tag{3.16}$$

Problems

Problem 3.1

(a) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence and let $\{y_n\}_{n=1}^{\infty}$ be a divergent sequence. What can be said of the produt sequence $\{x_n y_n\}_{n=1}^{\infty}$.

(b) If a sequence of integer numbers is convergent, what is this sequence like?

(c) Prove that every convergent sequence is bounded.

Problem 3.2 Given the following recurrent sequences, find the general term and compute their limit:

(i) $a_{n+1} = \dfrac{a_n + 1}{2}$, with $a_0 = 0$; (ii) $b_{n+1} = \sqrt{2b_n}$, with $b_0 = 1$.

Problem 3.3 Calculate the following limits:

(i) $\lim_{n\to\infty} \sqrt[n]{a^n + b^n}$, with $a, b > 0$;

(ii) $\lim_{n\to\infty} \left(\dfrac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n$, with $a, b > 0$;

(iii) $\lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right)$;

(iv) $\lim_{n\to\infty} \sqrt{n}\left(\sqrt[4]{n^2+1} - \sqrt{n+1}\right)$;

(v) $\lim_{n\to\infty} \dfrac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$;

(vi) $\lim_{n\to\infty} \left(\dfrac{n^2+1}{n^2 - 3n}\right)^{\frac{n^2-1}{2n}}$.

Problem 3.4 Calculate the following limits:

(i) $\lim_{n\to\infty} \dfrac{n}{\pi} \sin n\pi$;

(ii) $\lim_{n\to\infty} \dfrac{n\left(e^{1/n} - e^{\sin(1/n)}\right)}{1 - n\sin(1/n)}$;

(iii) $\lim_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}}$;

(iv) $\lim_{n\to\infty} n^{-3/n}$;

(v) $\lim_{n\to\infty} \dfrac{2^n}{n!}$;

(vi) $\lim_{n\to\infty} \dfrac{n^2}{2^n}$;

(vii) $\lim_{n\to\infty} \dfrac{n^{n-1}}{(n-1)^n}$;

(viii) $\lim_{n\to\infty} \dfrac{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \cdots + n\sqrt[n]{n}}{n^2}$.

Problem 3.5 If $a > 0$ and $\lim_{n\to\infty} u_n = 0$, calculate the following limits:

(i) $\lim_{n\to\infty} \left(\cos\dfrac{b}{n} + a\sin\dfrac{b}{n}\right)^n$; (ii) $\lim_{n\to\infty} \sqrt[u_n]{\dfrac{a - bu_n}{a + bu_n}}$.

Problem 3.6 Calculate the following limits:

(i) $\lim_{n\to\infty} \dfrac{\sum_{k=1}^{n} \sin(\pi/k)}{\log n}$; (ii) $\lim_{n\to\infty} \prod_{k=1}^{n} (2k-1)^{1/n^2}$; (iii) $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{k^2}{n^2} \sin\dfrac{1}{k}$.

Problem 3.7 Given that $\lim_{n\to\infty} a_n = a$, calculate

$$\lim_{n\to\infty} \frac{a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}}{\log(n+1)}.$$

Problem 3.8 Calculate the limit

$$\lim_{n\to\infty}\sum_{k=1}^{3n}\frac{1}{\sqrt{n^2+k}}$$

using the sandwich rule.
HINT: Use the largest and smallest terms in the sum to bound the sum from above and from below, respectively.

Problem 3.9 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_n-n)=\ell$.

(a) Prove that $\lim_{n\to\infty}\frac{a_n}{n}=1$.

(b) Prove that $\lim_{n\to\infty} n\log(a_n/n)=\ell$.

Problem 3.10 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_{n+1}/a_n)=\ell$. Apply Stolz theorem to calculate the limit

$$\lim_{n\to\infty}\sqrt[n^2]{\frac{a_n^n}{a_1a_2\cdots a_n}}$$

Problem 3.11 Prove that the following sequences are monotonic, determine whether they are bounded, and find the limit in case they are:

(i) $\sqrt{2},\sqrt{2\sqrt{2}},\sqrt{2\sqrt{2\sqrt{2}}},\dots;$

(ii) $\sqrt{2},\sqrt{2+\sqrt{2}},\sqrt{2+\sqrt{2+\sqrt{2}}},\dots;$

(iii) $u_{n+1}=3+\frac{u_n}{2}$, with $u_0=0$;

(iv) $u_{n+1}=3+2u_n$, with $u_0=0$;

(v) $u_{n+1}=\frac{u_n^3+6}{7}$, with (a) $u_0=1/2$, (b) $u_0=3/2$, and (c) $u_0=3$.

Problem 3.12 Consider the sequence defined by $a_{n+1}=\sqrt{1+3a_n}-1$, with $a_0=1/2$.

(a) Prove that the sequence has a limit and find it.

(b) Compute $\lim_{n\to\infty}\frac{a_{n+1}-1}{a_n-1}$.

Problem 3.13 Consider the sequence defined by $b_{n+1}=1-b_n/2$, with $b_0=0$.

(a) Prove that the sequence is alternating, i.e., $(b_{n+1}-b_n)(b_n-b_{n-1})<0$.

(b) Assuming that it has a limit ℓ, find it.

(c) Prove that $|b_{n+1}-\ell|=\frac{1}{2}|b_n-\ell|$.

(d) Prove that the sequence has indeed a limit.

Problem 3.14 Consider the sequence defined by

$$x_{n+1}=\frac{x_n(1+x_n)}{1+2x_n},\qquad x_1=1.$$

(a) Prove that $x_n>0$ for all $n\in\mathbb{N}$.

(b) Prove that the sequence is monotonically decreasing.

(c) Calculate its limit.