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Calculus I

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Unit 4. Series

4. Series

4.1 Series of real numbers

Series are a special kind of sequences —those made of sums of terms of other sequences. As sequences, they share all properties of sequences of real numbers studied in the previous chapter. However, studying the convergence of a series is normally a difficult task —not to mention to actually calculate its limit when it exists. That is why they are studied separately, and special techniques to address their convergence have to be developed.

Suppose we have a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$. With this sequence we can construct another sequence, that we will denote $\{S_k\}_{k=1}^{\infty}$, whose general term is the sum of all terms of the original sequence up to the nth, namely

$$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n. \tag{4.1}$$

(In what follows we will make common use of symbolic sums, whose properties can be found in Appendix A.) The general term S_k of this new sequence is often referred to as the kth **partial sum** of the original sequence.

The limit of the sequence of partial sums has a special notation:

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty} a_n. \tag{4.2}$$

This "infinite sum" (actually a limit) is what we normally refer to as a **series.**

■ **Example 4.1** As a first illustrative example we will introduce the **geometric series**

$$\sum_{n=0}^{\infty} x^n, \qquad x \in \mathbb{R}, \tag{4.3}$$

in other words, out of the sequence $\{x^n\}_{n=0}^{\infty}$ we construct the sequence of partial sums

$$S_k = \sum_{n=0}^{k} x^n = 1 + x + x^2 + \cdots + x^k.$$

The geometric series is the limit of this sequence.

The geometric series is one of the very few cases in which not only its convergence can be fully characterised, but also the sum can be explicitly computed when it converges. The reason is that an alternative expression for S_n can be obtained.

This is achieved by realising that if we multiply S_n by x we almost recover S_n again:

$$xS_k = \underbrace{x + x^2 + \cdots + x^k}_{=S_k - 1} + x^{k+1} = S_k - 1 + x^{k+1} \qquad \Rightarrow \qquad (x-1)S_k = x^{k+1} - 1.$$

So if $x \neq 1$, we obtain

$$S_k = \frac{x^{k+1}-1}{x-1} = \frac{1-x^{k+1}}{1-x},$$

and if $x = 1$ clearly $S_k = k+1$. In summary,

$$S_k = \begin{cases} \dfrac{1-x^{k+1}}{1-x}, & x \neq 1, \\ k+1 & x = 1. \end{cases} \tag{4.4}$$

In terms of x we can distinguish these cases:

(a) If $x = 1$ then $S_k = k+1$ diverges to $+\infty$.

(b) If $|x| < 1$ then $\lim_{k\to\infty} x^{k+1} = 0$

$$\lim_{k\to\infty} S_k = \frac{1}{1-x}.$$

(c) If $x > 1$ then $\lim_{k\to\infty} x^{k+1} = +\infty$ and S_k diverges to $+\infty$.

(d) If $x < -1$ then x^k is an alternating sequence with no limit.

All this information is usually summarised in the formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1. \tag{4.5}$$

■

This example also illustrate the three cases we can meet when we address the convergence of a series:

(a) S_n converges; then we say that $\sum_{n=1}^{\infty} a_n$ is **convergent.**

(b) S_n diverges to $\pm\infty$; then we say that $\sum_{n=1}^{\infty} a_n$ is **divergent.**

(c) S_n has no limit (e.g., is alternating); then we say that $\sum_{n=1}^{\infty} a_n$ is **not convergent.**

R The convergence of a series is not affected by altering (adding, removing, changing...) a *finite* number of its terms. However, if it converges, the sum does change.

■ **Example 4.2** The convergence of the series

$$\sum_{n=r}^{\infty} x^n$$

in terms of x is exactly the same as that of the geometric series. The removal of the first r terms in the latter does not affect its character. In this case it is particularly evident because

$$\sum_{n=r}^{\infty} x^n = \sum_{n=0}^{\infty} x^{r+n} = x^r \sum_{n=0}^{\infty} x^n,$$

so both series are proportional to each other. In particular this implies

$$\sum_{n=r}^{\infty} x^n = \frac{x^r}{1-x}, \qquad |x| < 1, \tag{4.6}$$

so the sum is different. ■

■ **Example 4.3** A small variation of the geometric series is the arithmetic-geometric series. It is defined as

$$\sum_{n=1}^{\infty} nx^n.$$

(Starting at $n = 1$ or $n = 0$ is irrelevant because for $n = 0$ the corresponding term $nx^n = 0$.) We proceed through a similar argument. Let

$$S_k = \sum_{n=1}^{k} nx^n = x + 2x^2 + 3x^3 + \cdots + (k-1)x^{k-1} + kx^k.$$

To begin with, if $x = 1$ then

$$S_k = 1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

In this case $\lim_{k\to\infty} S_k = \infty$.

Suppose now that $x \neq 1$ and multiply S_k by x. Then

$$xS_k = x^2 + 2x^3 + 3x^4 + \cdots + (k-1)x^k + kx^{k+1}.$$

Then, substracting

$$\begin{aligned} S_k - xS_k &= \left(x + 2x^2 + 3x^3 + \cdots + kx^k\right) - \left(x^2 + 2x^3 + \cdots + (k-1)x^k + kx^{k+1}\right) \\ &= x + x^2 + x^3 + \cdots + x^k - kx^{k+1}. \end{aligned}$$

The positive terms in the right-hand side form the partial sum of the geometric —except for the first term, which is missing. Then

$$(1-x)S_k = \frac{x^{k+1}-1}{x-1} - 1 - kx^{k+1} = \frac{x^{k+1} - x - kx^{k+2} + kx^{k+1}}{x-1} = \frac{x - (k+1)x^{k+1} + kx^{k+2}}{1-x}$$

and therefore

$$S_k = \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2}.$$

Now, if $x > 1$ the partial sum S_k diverges when $k \to \infty$ because of the term kx^{k+2}. If $x < -1$ the partial sum S_k does not have a limit when $k \to \infty$ because kx^{k+2} alternates sign and grows indefinitely in size. Finally, if $|x| < 1$ then

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2} = \frac{x}{(1-x)^2}.$$

All this is summarized in the equation

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \qquad |x| < 1. \tag{4.7}$$

■

Exercise 4.1 Using a similar procedure, prove that

$$(1-x)\sum_{n=1}^{k} n^2x^n = 2\sum_{n=1}^{k} nx^n - \sum_{n=1}^{k} x^n - k^2x^{k+1}.$$

By taking the limit in this expression, finally prove that

$$\sum_{n=1}^{\infty} n^2x^n = \frac{x+x^2}{(1-x)^3}, \qquad |x|<1. \tag{4.8}$$

■

Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series. This means that $\lim_{n\to\infty} S_n$ exists. But clearly $a_n = S_n - S_{n-1}$, therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0.$$

In other words:

Proposition 4.1.1 If $\sum_{n=1}^{\infty} a_n$ is a convergent series then $\lim_{n\to\infty} a_n = 0$.

Or in a form that turns out to be more useful: if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge (it either diverges or not converges).

■ **Example 4.4** The series

$$1-1+1-1+1-1+\cdots = \sum_{n=0}^{\infty}(-1)^n$$

does not converge because $a_n = (-1)^n$ does not tend to zero as $n\to\infty$ (as a matter of fact, it does not even converge because it is alternating).

This is another case in which all can be told from the expresion of the partial sum, because

$$S_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

So S_k itself is alternating, and therefore not convergent. ■

■ **Example 4.5** The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$$

does not converge (in fact it diverges to $+\infty$) because

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0.$$

■

Unfortunately the converse of Proposition 4.1.1 is not true (otherwise telling the convergence of a series would be a trivial matter), as the following example illustrates:

■ **Example 4.6**

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

The reason is that $S_k \sim \log k$, as we know from Example 3.19. However, as we can see,

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

■

4.2 Series of nonnegative terms

A **series of nonnegative terms** is a series $\sum_{n=1}^{\infty} a_n$ such that $a_n \geqslant 0$ for all $n \in \mathbb{N}$.

What is most relevant of this kind of series is that *the sequence of partial sums is monotically increasing,* because $S_k - S_{k-1} = a_k \geqslant 0$. Therefore either the partial sums are bounded above —and then the series converges— or they are unbounded —and then the series diverges to $+\infty$.

Because of this property, every subsequence of $\{S_k\}_{k=1}^{\infty}$ will also be monotically increasing and convergent to the same limit (or divergent it the series diverges). In other words: in this kind of series *convergence can be decided on any subsequence of partial sums.* Often this simplifies the problem.

A further consequence is that series, interpreted as "infinite sums", satisfy the conmutative and associative properties. This is sucintly captured by the term *inconditional convergence.* Series of nonnegative terms are inconditionally convergent.

Based on these facts there is a set of tests to check convergence of a series of nonnegative terms. Of the very many that can be found in the literature, we will simply list here the most common ones.

Theorem 4.2.1 — Comparison test. Let $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} b_n < \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} a_n < \infty.$$

Alternatively,

$$\sum_{n=1}^{\infty} a_n = \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} b_n = \infty.$$

Proof. Since $\sum_{n=1}^{k} a_n \leqslant \sum_{n=1}^{k} b_n \leqslant \sum_{n=1}^{\infty} b_n < \infty$, the sequence of partial sums of $\{a_n\}_{n=1}^{\infty}$ is bounded above. ■

The requirement $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$ is too strict. We can relax it to $0 \leqslant a_n \leqslant b_n$ for all $n > N$ (for some $N \in \mathbb{N}$). The reason is that what happens to a finite number of terms (the first N ones) is irrelevant for the convergence of the series.

Exercise 4.2 Show that $\sum_{n=0}^{\infty} \frac{1}{n!} < \infty$ by comparing this series with the geometric series. (Note that $n! > 2^n$ for $n > 3$.) ■

Exercise 4.3 Show that $\sum_{n=2}^{\infty} \frac{1}{\log n} = \infty$ by comparing this series with the harmonic series. ■

Theorem 4.2.2 — Condensation test. Let $\{a_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of nonnegative terms, and let $q_0 < q_1 < \cdots < q_k < \cdots$ be a strictly increasing sequence of natural numbers. Then

$$\sum_{k=0}^{\infty}(q_{k+1}-q_k)a_{q_k} < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < \infty \quad \text{and} \quad \sum_{k=0}^{\infty}(q_{k+1}-q_k)a_{q_{k+1}} = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = \infty.$$

Proof. We can split the partial sum $S_{q_{p+1}} = \sum_{n=1}^{q_{p+1}} a_n$ in blocks as

$$S_{q_{p+1}} = S_{q_0} + (S_{q_1} - S_{q_0}) + (S_{q_2} - S_{q_1}) + \cdots + (S_{q_{p+1}} - S_{q_p}).$$

Then

$$S_{q_{k+1}} - S_{q_k} = a_{q_k+1} + a_{q_k+2} + \cdots + a_{q_{k+1}} \leqslant (q_{k+1} - q_k)a_{q_k}$$

because the sequence a_n is monotonically decreasing and thus a_{q_k} is an upper bound to all terms in the sum. Therefore

$$S_{q_{p+1}} \leqslant S_{q_0} + \sum_{k=0}^{p}(q_{k+1} - q_k)a_{q_k},$$

so $\sum_{k=0}^{\infty}(q_{k+1} - q_k)a_{q_k} < \infty$ implies $\sum_{n=1}^{\infty} a_n < \infty$ because of the comparison test. Likewise, as $a_{q_{k+1}}$ is a lower bound to all terms in the sum, $S_{q_{k+1}} - S_{q_k} \geqslant (q_{k+1} - q_k)a_{q_{k+1}}$, so

$$S_{q_0} + \sum_{k=0}^{p}(q_{k+1} - q_k)a_{q_{k+1}} \leqslant S_{q_{p+1}}$$

and hence $\sum_{k=0}^{\infty}(q_{k+1} - q_k)a_{q_{k+1}} = \infty$ implies $\sum_{n=1}^{\infty} a_n = \infty$, again as a consequence of the comparison test. ■

As a corollary to this theorem we can obtain a simpler version that commonly appears in textbooks as *Cauchy's condensation test*. It just amounts to taking the sequence of integers $q_k = 2^k$ in the previous theorem. As $q_{k+1} - q_k = 2^k$, then $(q_{k+1} - q_k)a_{q_k} = 2^k a_{2^k}$ and $(q_{k+1} - q_k)a_{q_{k+1}} = 2^k a_{2^{k+1}} = \frac{1}{2}2^{k+1}a_{2^{k+1}}$. Hence both "condensed" series are one and the same. As a matter of fact, the choice of 2 is made for purely historical reasons because any other natural number $m > 1$ will do—notice that $m^{k+1} - m^k = (m-1)m^k = m^{-1}(m-1)m^{k+1}$.

Corollary 4.2.3 — Cauchy's condensation test. Let $\{a_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of nonnegative terms. Then

$$\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} a_n < \infty.$$

■ **Example 4.7** We will apply Cauchy's condensation test to decide the convergence of Riemann's (or the generalised harmonic) series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}.$$

To this purpose we compute

$$\sum_{k=0}^{\infty} \frac{2^k}{2^{k\alpha}} = \sum_{k=0}^{\infty} \frac{1}{2^{k(\alpha-1)}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^k.$$

This is the geometric series, which we know is convergent if and only if

$$\frac{1}{2^{\alpha-1}} < 1 \qquad \Leftrightarrow \qquad 2^{\alpha-1} > 1 \qquad \Leftrightarrow \qquad \alpha > 1.$$

Therefore Riemann's series converges if and only if $\alpha > 1$. ■

■ **Example 4.8** Cauchy's condensation test is particularly useful when logarithms are involved. For instance, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$$

and compute

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k (k \log 2)^{\alpha}} = \frac{1}{(\log 2)^{\alpha}} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}},$$

which, up to a constant, is Riemann's series. Therefore the tested series converges if and only if $\alpha > 1$. ■

One of the most powerful tests is the *limit comparison test.* The idea behind it is that two series that behave similarly as $n \to \infty$ have the same convergence properties. So the notions of equivalent and negligible sequences adquire a special relevance here.

Theorem 4.2.4 — Limit comparison test. Given the series of nonnegative terms $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$:

(a) If $a_n \sim b_n$ then

$$\sum_{n=1}^{\infty} a_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n = \infty.$$

(b) If $a_n \ll b_n$ then

$$\sum_{n=1}^{\infty} b_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n = \infty.$$

Proof.

(a) $a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Thus, according to the definition of limit, given $\varepsilon > 0$,

$$-\varepsilon < \frac{a_n}{b_n} - 1 < \varepsilon \quad \Leftrightarrow \quad 1-\varepsilon < \frac{a_n}{b_n} < 1+\varepsilon \quad \Leftrightarrow \quad (1-\varepsilon)b_n < a_n < (1+\varepsilon)b_n$$

for large enough n. If we take $\varepsilon < 1$, (a) follows from the last inequality by the comparison test.

(b) $a_n \ll b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Thus, given $\varepsilon > 0$,

$$-\varepsilon < \frac{a_n}{b_n} < \varepsilon \quad \Leftrightarrow \quad -\varepsilon b_n < a_n < \varepsilon b_n$$

for large enough n. Hence (b) follows from $a_n < \varepsilon b_n$ by the comparison test (the inequality $-\varepsilon b_n < a_n$ is true but useless).

■

■ **Example 4.9** In order to know if the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^2+2n+7}}$$

converges or diverges, all we need to know is that $3n^2+2n+7 \sim 3n^2$, so $\sqrt{3n^2+2n+7} \sim \sqrt{3}n$. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3}n} = \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

because it is the harmonic series, then the series we are testing also diverges. ■

Exercise 4.4 Prove, using appropriately the limit comparison test, that if $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} n^{\alpha} a_n = \ell \geqslant 0,$$

then $\sum_{n=1}^{\infty} a_n < \infty$ if $\alpha > 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ if $\alpha \leqslant 1$. ■

Theorem 4.2.5 — Root test. If $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \ell \geqslant 0,$$

then $\sum_{n=1}^{\infty} a_n < \infty$ if $\ell < 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ if $\ell > 1$. (The case $\ell = 1$ remains undecided.)

Proof. Given $\ell > \varepsilon > 0$,

$$-\varepsilon < \sqrt[n]{a_n} - \ell < \varepsilon \quad \Leftrightarrow \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon \quad \Leftrightarrow \quad (\ell-\varepsilon)^n < a_n < (\ell+\varepsilon)^n$$

for large enough n. Now, if $\ell < 1$ we can take ε so that $\ell + \varepsilon < 1$ (e.g., $\varepsilon = (1-\ell)/2$). Therefore the geometric series

$$\sum_{n=1}^{\infty} (\ell+\varepsilon)^n < \infty$$

and $\sum_{n=1}^{\infty} a_n < \infty$ by the comparison test. On the contrary, if $\ell > 1$ we can take ε so that $\ell - \varepsilon > 1$ (e.g., $\varepsilon = (\ell-1)/2$). Hence

$$\sum_{n=1}^{\infty} (\ell+\varepsilon)^n = \infty$$

and $\sum_{n=1}^{\infty} a_n = \infty$ by the comparison test.

If $\ell = 1$ neither inequality is useful and we can conclude nothing. ■

Corollary 3.4.4 to Stolz's theorem transforms this test into another one:

Corollary 4.2.6 — Quotient test. If $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \ell \geqslant 0,$$

then $\sum_{n=1}^{\infty} a_n < \infty$ if $\ell < 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ if $\ell > 1$. (The case $\ell = 1$ remains undecided.)

4.3 Alternating series

When the terms of a series can be either positive or negative, the test developed in the previous section are no longer valid. However, these series can be classified into two main groups according to whether they are *absolutely convergent* or not.

Definition 4.3.1 — Absolutely converget series. $\sum_{n=1}^{\infty} a_n$ is said to be an absolutely convergent series if $\sum_{n=1}^{\infty} |a_n| < \infty$.

It is easy to prove that absolutely convergent series are also convergent in the usual sense. Therefore all tests for series of nonnegative terms can be applied to the series of absolute values.

Dirichlet proved that absolutely convergent series converge inconditionally. On the contrary, series that do not converge absolutely are conditionally convergent. This means that a permutation and/or association of their terms can change their sum (this result was proven by Riemann). Their interpretation as "infinite sums" is thus weaker than that of absolutely convergent series, because an "order of sum" must be specified in advance (as a matter of fact, this is what the definition of a convergent series does).

■ **Example 4.10** Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

We will show later, in Example 4.11, that this series converges, but clearly does not do it absolutely (because the series of absolute values is the harmonic series). Let us denote S its sum. Then

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Let us reorder its terms by choosing one positive followed by two negative terms, in order, and then associate each positive with the first subsequent negative. We obtain

$$\begin{aligned} S' &= \underbrace{\left(1 - \frac{1}{2}\right)}_{=1/2} - \frac{1}{4} + \underbrace{\left(\frac{1}{3} - \frac{1}{6}\right)}_{=1/6} - \frac{1}{8} + \underbrace{\left(\frac{1}{5} - \frac{1}{10}\right)}_{=1/10} - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{S}{2}, \end{aligned}$$

so with this manipulation the series sums half its initial value. ■

Among the series with arbitrary sign patterns the most frequently met are those with alternating signs. These are referred to as **alternating series.** They can have either of the two forms

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n, \qquad \sum_{n=1}^{\infty} (-1)^n a_n, \tag{4.9}$$

where $a_n \geqslant 0$.

R Note that $\sum_{n=1}^{\infty}(-1)^n a_n = -\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, so the two forms are actually equivalent.

There is only one test for alternating series:

Theorem 4.3.1 — Leibniz test. If $\{a_n\}_{n=1}^{\infty}$ decreases monotonically and $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \tag{4.10}$$

converges.

■ **Example 4.11** The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges according to Leibniz's test because $a_n = 1/n$ is a monotonically decreasing sequence that approaches 0 as $n \to \infty$. ■

4.4 Telescoping series

A series $\sum_{n=1}^{\infty} a_n$ is said to be **telescoping** if there exists a sequence $\{u_n\}_{n=1}^{\infty}$ such that $a_n = u_n - u_{n+1}$ for all $n \in \mathbb{N}$.

The importance of telescoping series is that they can be easily summed. The reason is that the partial sum

$$S_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k}(u_n - u_{n+1}) = u_1 - u_{k+1},$$

therefore we have the simple formula

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(u_n - u_{n+1}) = u_1 - \lim_{n\to\infty} u_n. \tag{4.11}$$

■ **Example 4.12** Consider the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$$

Since we can expand

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

the series telescopes identifying $u_n = 1/n$, so

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty}\frac{1}{n} = 1.$$

■

■ **Example 4.13** Consider the series

$$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right).$$

Since

$$\log\left(1+\frac{1}{n}\right) = \log\left(\frac{n+1}{n}\right) = \log(n+1) - \log n$$

the series telescopes identifying $u_n = -\log n$, so

$$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right) = 0 + \lim_{n\to\infty} \log n = \infty.$$

The series diverges to $+\infty$. ■

■ **Example 4.14** Consider the series

$$\sum_{n=1}^{\infty} \arctan\frac{1}{1+n+n^2}.$$

The key to show that this series telescopes is to realise that

$$\frac{1}{1+n+n^2} = \frac{1}{1+n(n+1)} = \frac{(n+1)-n}{1+n(n+1)},$$

and to compare this formula with the trigonometric identity

$$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y},$$

which allows us to indentify $x = \arctan(n+1)$ and $y = \arctan n$. In other words,

$$\arctan\frac{1}{1+n+n^2} = \arctan\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \arctan(n+1) - \arctan n$$

and the series telescopes identifying $u_n = -\arctan n$. Therefore

$$\sum_{n=1}^{\infty} \arctan\frac{1}{1+n+n^2} = -\arctan 1 + \lim_{n\to\infty} \arctan n = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}.$$

■

To be honest, every series is telescoping because

$$S_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} (u_n - u_{n+1}) = u_1 - u_{k+1},$$

so we can identify $u_k \equiv u_1 - S_{k-1}$, and choose u_1 arbitrarily. The problem is that this is tantamount to being able to calculate S_k —usually a difficult problem. So we really call telescoping those series for which the identification $a_n = u_n - u_{n+1}$ is more or less explicit —as in the previous examples.

Problems

Problem 4.1 Determine the convergent character of the following series of nonnegative terms:

(i) $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)^n$;

(ii) $\sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}$;

(iii) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n^4+1}}$;

(iv) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$;

(v) $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2+n}$;

(vi) $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$;

(vii) $\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}$;

(viii) $\sum_{n=1}^{\infty}\frac{3n-1}{(\sqrt{2})^n}$;

(ix) $\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}$;

(x) $\sum_{n=1}^{\infty}(\sqrt[n]{n}-1)^n$;

(xi) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}3^{-n}$;

(xii) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$;

(xiii) $\sum_{n=2}^{\infty}\frac{n^2}{(\log n)^n}$;

(xiv) $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$;

(xv) $\sum_{n=2}^{\infty}\frac{1}{n^{\log n}}$;

(xvi) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}$;

(xvii) $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$;

(xviii) $\sum_{n=2}^{\infty}\left(\frac{n}{n-1}\right)^n$.

Problem 4.2 Prove that the series

$$\sum_{n=1}^{\infty}\left(\frac{a}{2n-1}-\frac{b}{2n+1}\right)$$

converges if, and only if, $a=b$, and in that case calculate its sum.

Problem 4.3 Discuss, depending on the value of the parameter a in the given range, whether the following series converge or diverge:

(i) $\sum_{n=1}^{\infty}n(1+a)^n e^{-na}$, for $a>-1$;

(ii) $\sum_{n=1}^{\infty}\frac{n^n}{a^n n!}$, for $a>0$;

(iii) $\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+a}}$, for any $a\in\mathbb{R}$;

(iv) $\sum_{n=1}^{\infty}\frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$, for $a\geqslant 0$.

Problem 4.4 Determine whether the following series are absolutely convergent, and if not, whether they converge conditionally:

(i) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$;

(ii) $\sum_{n=1}^{\infty}\sin\left(\pi n+\frac{1}{n}\right)$;

(iii) $\sum_{n=1}^{\infty}(-1)^n\left(\arctan\frac{1}{n}\right)^2$;

(iv) $\sum_{n=1}^{\infty}(-1)^n(\arctan n)^2$;

(v) $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n^2-1}-n\right)$;

(vi) $\sum_{n=1}^{\infty}(-1)^n\log\left(\frac{n}{n+1}\right)$;

(vii) $\sum_{n=1}^{\infty}(-1)^n\left(1-\cos\frac{1}{n}\right)$;

(viii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\log(e^n+e^{-n})}$.

Problem 4.5 Sum the following series:

(i) $\sum_{n=0}^{\infty} \frac{3^{n+1} - 2^{n-3}}{4^n}$;

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$;

(iii) $\sum_{n=0}^{\infty} \frac{4n+1}{3^n}$;

(iv) $\sum_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right)$;

(v) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}}$;

(vi) $\sum_{n=1}^{\infty} \log\left[\frac{n(n+2)}{(n+1)^2}\right]$;

(vii) $\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor}$, with $|xy| < 1$;

(viii) $\sum_{n=0}^{\infty} \frac{1}{2^n} \cos\frac{2\pi n}{3}$.

Problem 4.6 Obtain the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n^2 + 2n}$ by rewriting it as a telescoping series.
HINT: Expand the general term in elementary fractions.

Problem 4.7 Let $\mathscr{C}_0$ be a circle of radius r. Let $\mathscr{Q}_0$ be a square inscribed in $\mathscr{C}_0$. Let $\mathscr{C}_1$ be the circle inscribed in $\mathscr{Q}_0$, and $\mathscr{Q}_1$ a square inscribed in $\mathscr{C}_1$. Continue the process this way and obtain the sequence of circles $\{\mathscr{C}_n\}_{n=0}^{\infty}$ with radii $\{r_n\}_{n=0}^{\infty}$. What is the sum of the areas of these infinitely many circles?

Problem 4.8 Calculate $\lim_{n\to\infty} a_n$, where $a_n = \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\cdots\sqrt[2^n]{2}$.
HINT: Calculate the limit of $\log a_n$ first.

Problem 4.9 Let $b_0 \in \mathbb{Z}$, $b_n \in \{0,1,2,\ldots,9\}$, for $n = 1,2,\ldots$, and form the series

$$\sum_{n=0}^{\infty} \frac{b_n}{10^n}.$$

(a) Prove that this series converges.

(b) Discuss the meaning of this series and why it is so important.

(c) Calculate its sum for $b_n = 9$ for all $n \geqslant 0$.

(d) Calculate its sum if $b_n = 1$ for n even and $b_n = 2$ for n odd.

Problem 4.10

(a) Prove (graphically or otherwise) that the equation $\tan x = x$ has a solution $(2n-1)\frac{\pi}{2} < \lambda_n < (2n+1)\frac{\pi}{2}$ for every $n \in \mathbb{N}$.

(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$.

Problem 4.11

(a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ two convergent series of nonnegative terms. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n b_n} < \infty$.
HINT: Use the inequality $xy \leqslant (x^2 + y^2)/2$.

(b) As an application prove that if the series of nonnegative terms $\sum_{n=1}^{\infty} a_n < \infty$ then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty$.

Problem 4.12 Let $\{u_n\}_{n=1}^{\infty}$ be the sequence of all positive integers containing *no zeros* in their decimal expression.

(a) Prove that $\sum_{n=1}^{\infty} \frac{1}{u_n} < 90$.
HINT: Group all terms u_n with the same number of decimal digits.

(b) What can you say about the series $\sum_{n=1}^{\infty} \frac{1}{w_n}$, where $\{w_n\}_{n=1}^{\infty}$ is the sequence of all positive integers containing *at least one zero* in their decimal expression?

Problem 4.13 In a real *tour-de-force* we are going to calculate the —apparently impossible— sum of the conditionally convergent series

$$\sum_{n=1}^{\infty}(-1)^n \log\left(\frac{n}{n+1}\right).$$

We will do that in steps:

(a) Show that

$$2\cdot 4\cdot 6\cdots(2n) = n!2^n, \qquad 1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{n!2^n}.$$

(b) Use Stirling to prove

$$2\cdot 4\cdot 6\cdots(2n) \sim \sqrt{2\pi n}e^{-n}(2n)^n, \qquad 1\cdot 3\cdot 5\cdots(2n-1) \sim \sqrt{2}e^{-n}(2n)^n.$$

(c) Show that the partial sum S_{2k} of the series above can be written

$$S_{2k} = \log\left(\frac{2^2\cdot 4^2\cdot 6^2\cdots(2k)^2}{1\cdot 3^2\cdot 5^2\cdots(2k-1)^2(2k+1)}\right).$$

(d) Use the Stirling formulas derived above to calculate the limit of S_{2k} when $k\to\infty$. Why does this provide the answer to the problem?

Problem 4.14 Suppose a certain series can be written as

$$\sum_{n=1}^{\infty}(\alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2}), \qquad \alpha_0+\alpha_1+\alpha_2 = 0.$$

(a) Rewrite the general term as an ordinary telescoping series and provide a formula for the sum.

(b) Apply this result to calculate the sum

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)(n+2)}.$$

(c) Do the same for the general case

$$\sum_{n=1}^{\infty}(\alpha_0 u_n + \alpha_1 u_{n+1} + \cdots + \alpha_k u_{n+k}), \qquad \sum_{j=0}^{k}\alpha_j = 0.$$

HINT: In (a), add and substract $\alpha_0 u_{n+1}$ and replace $\alpha_2 = -(\alpha_0+\alpha_1)$. Use a similar procedure in (c).