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Calculus I

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Unit 5. Limit of a Function

5. Limit of a Function

5.1 Concept and definition

Functions are defined for every single point of their domains. However, differential calculus has to do with the behaviour of functions "around" points, not just at them. The limit of a function is a way to characterise that behavior. The idea is to know what value the function is approaching as we get closer and closer to a certain point a (not necessarily in the domain of the function).

Our first definition will base this knowledge in the well-known limit of sequences.

Definition 5.1.1 We say that the **limit of a function** $f : A \to \mathbb{R}$ when x approaches a is ℓ, and denote it

$$\lim_{x \to a} f(x) = \ell, \tag{5.1}$$

if, *for every sequence* $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \neq a$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} x_n = a,$$

it holds

$$\lim_{n \to \infty} f(x_n) = \ell.$$

Having a sequence in the domain that tends to a as $n \to \infty$ is a way to approach a. The condition $x_n \neq a$ is there to account for those cases in which $a \notin A$. Note the remark "for every sequence" in the definition. It is very important because if it holds, then what $f(x)$ tends to does not depend on how we approach a.

■ **Example 5.1** Consider the function $f(x) = x^2$ and the point $a = 2$ (in the domain). Take the sequence $x_n = 2 + \varepsilon_n$. As long as $\varepsilon_n \neq 0$ for every $n \in \mathbb{N}$ and $\varepsilon_n \to 0$ as $n \to \infty$, this sequence satisfies the conditions of the definition. Then

$$f(x_n) = (2 + \varepsilon_n)^2 = 4 + 2\varepsilon_n + \varepsilon_n^2 \xrightarrow[n \to \infty]{} 4.$$

This proves that

$$\lim_{x\to 2} x^2 = 4.$$

■

■ **Example 5.2** The previous example might suggest that calculating a limit could be as simple as evaluating $f(a)$. To show that this is not always the case consider the function

$$f(x) = \frac{x-1}{x^2-1},$$

a rational function whose domain is $\mathbb{R} - \{1\}$. Take any sequence $x_n = 1 + \varepsilon_n$, with $\varepsilon_n \neq 0$. If $\varepsilon_n \to 0$ as $n \to \infty$ then $x_n \to 1$. Now,

$$f(x_n) = \frac{\varepsilon_n}{(1+\varepsilon_n)^2 - 1} = \frac{\varepsilon_n}{2\varepsilon_n + \varepsilon_n^2} = \frac{1}{2+\varepsilon_n} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

This proves that

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

even though 1 is *not* in the domain of f (hence $f(1)$ does not even exists). ■

■ **Example 5.3** For a final illustrating example consider the function $f(x) = \sin(\pi/x)$, whose domain is $\mathbb{R} - \{0\}$, and take $a = 0$ (not in the domain). Consider the sequence $x_n = 1/n$, satisfying the requirements of the definition. Now,

$$f(x_n) = \sin(\pi n) = 0$$

for all $n \to \infty$. So, if the limit exists, it has to be 0.

But now consider the sequence $y_n = 2/(4n+1)$. Then,

$$f(y_n) = \sin\left(\pi \frac{4n+1}{2}\right) = \sin\left(\pi\left(2n + \frac{1}{2}\right)\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1.$$

So we have found two different sequences, x_n and y_n, such that the two limits, $f(x_n)$ and $f(y_n)$ exist, but are different. So $f(x)$ does not have a limit when $x \to 0$. ■

Checking that the limit exists for every conceivable sequence might be a daunting task. For this reason there is this alternative (but equivalent) definition of limit, which is not based on sequences and is more widely used.

Definition 5.1.2 — ε-δ definition. We say that the **limit of a function** $f : A \to \mathbb{R}$ when x approaches a is ℓ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $0 < |x - a| < \delta$.

The idea of this definition is that for every $\varepsilon > 0$ you can find a $\delta > 0$ such that $(a-\delta, a) \cup (a, a+\delta) \subset f^{-1}\big((\ell - \varepsilon, \ell + \varepsilon)\big)$ (see Figure 5.1).

As with the limit of sequences (and for the same reason), if the limit exists it is unique. In other words, if the limit of $f(x)$ when $x \to a$ is both ℓ and m, then $\ell = m$.

5.1.1 One-sided limits

There is a difference between the limit when $n \to \infty$ and the limit when $x \to a$: in the former case we can only "approach ∞ from the left", whereas in the latter case we can approach a both from the left ($x < a$) or from the right ($x > a$). This motivate the definition of one-sided limit.

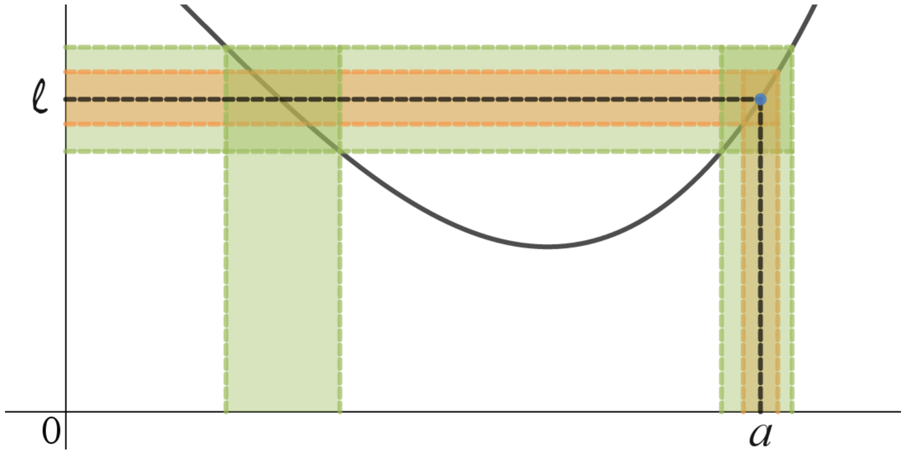

Figure 5.1: Sketch for the ε-δ definition of limit. The horizontal green stripe marks the interval $(\ell-\varepsilon,\ell+\varepsilon)$, so the two vertical stripes correspond to $f^{-1}\big((\ell-\varepsilon,\ell+\varepsilon)\big)$ —the set of points whose image is $(\ell-\varepsilon,\ell+\varepsilon)$. It is pictorially obvious that, no matter how narrow is the band, we can always construct an interval $(a-\delta,a+\delta)$ (vertical orange stripe) such that its image through f (except maybe that of a itself) (horizontal orange stripe) is contained in $(\ell-\varepsilon,\ell+\varepsilon)$.

Definition 5.1.3 — One-sided limit. We say that the **left-handed limit of a function** $f: A \to \mathbb{R}$ when x approaches a is ℓ, and denote it

$$\lim_{x\to a^-} f(x) = \ell, \tag{5.2}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $x_n < a$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

Similarly, we say that the **right-handed limit of a function** $f: A \to \mathbb{R}$ when x approaches a is ℓ, and denote it

$$\lim_{x\to a^+} f(x) = \ell, \tag{5.3}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $x_n > a$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

■ **Example 5.4** The Heaviside step function is defined as

$$H(x) = \begin{cases} 0, & x<0, \\ 1, & x>0, \end{cases} \tag{5.4}$$

and $H(0)$ is defined arbitrarily —if at all. Clearly for this function

$$\lim_{x\to 0^-} H(x) = 0, \qquad \lim_{x\to 0^+} H(x) = 1.$$

The two one-sided limits exist, but they are different. Clearly then $H(x)$ has no limit when $x \to 0$. ■

Proposition 5.1.1

$$\lim_{x\to a} f(x) = \ell \qquad \Leftrightarrow \qquad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell.$$

5.1.2 Infinite limits

We can adapt the definitions above to describe a function that grows unbounded when $x \to a$. For instance:

Definition 5.1.4 — Infinite limits. We say that the limit of a function $f : A \to \mathbb{R}$ when x approaches a is $+\infty$ (respectively $-\infty$), and denote it

$$\lim_{x\to a} f(x) = +\infty \qquad \text{(respectively } \lim_{x\to a} f(x) = -\infty\text{)}, \tag{5.5}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $x_n \neq a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = +\infty \qquad \text{(respectively } \lim_{n\to\infty} f(x_n) = -\infty\text{)}.$$

And similarly for the one-sided limits.

■ **Example 5.5** The function $f(x) = \frac{1}{|x|}$ tends to $+\infty$ when $x \to 0$. The reason is that for every sequence $x_n \neq 0$ such that $\lim_{n\to\infty} x_n = 0$,

$$\lim_{n\to\infty} \frac{1}{|x_n|} = \infty.$$

For the same reason, the function $f(x) = \frac{1}{x}$ has a right-handed limit $+\infty$ when $x \to 0^+$. However, its left-handed limit is $-\infty$, because taking any sequence $x_n < 0$ such that $\lim_{n\to\infty} x_n = 0$,

$$\lim_{n\to\infty} \frac{1}{x_n} = -\infty.$$

Thus, $1/x$ has no limit —not even infinite— when $x \to 0$. ■

Two particularly important one-sided limits are:

Definition 5.1.5 — Limit at the infinities. We say that the limit of a function $f : A \to \mathbb{R}$ when x approaches $+\infty$ (respectively $-\infty$) is ℓ, and denote it

$$\lim_{x\to+\infty} f(x) = \ell \qquad \text{(respectively } \lim_{x\to-\infty} f(x) = \ell\text{)}, \tag{5.6}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $\lim_{n\to\infty} x_n = +\infty$ (respectively $-\infty$), we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

Exercise 5.1 Extend the definitions above to express the cases of a function that approaches $\pm\infty$ when x approaches $\pm\infty$ (all combinations of signs). ■

A function such that $f(x) \to \pm\infty$ when $x \to a^{\pm}$ is said to *diverge* at $x = a$.

■ **Example 5.6** The function $f(x) = \frac{1}{x} \to 0$ when $x \to \pm\infty$. Let us see it for the case $x \to +\infty$. Take any sequence x_n that diverges to $+\infty$. Then

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

For $x \to -\infty$ the argument is similar. ■

5.2 Algebraic properties

As in the case of sequences, computing limits through the definition is not an easy task. However —as for sequences— limits satisfy a set of properties that allow us to do algebraic manipulations with limits and simplify their calculations.

Proposition 5.2.1 Let f and g be two real functions such that $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = \ell'$. Then the following properties hold:

1. $\lim_{x\to a} \left[f(x) \pm g(x)\right] = \ell + \ell'$;
2. $\lim_{x\to a} f(x)g(x) = \ell\ell'$;
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{\ell'}$ provided $\ell' \neq 0$.

(These properties hold also in the case $a = \pm\infty$ or even for one-sided limits.)

Proposition 5.2.2 Let f a real function such that $\lim_{x\to a} f(x) = \ell$. Then the following properties hold:

1. f is bounded in an evironment of a;
2. if $\ell \neq 0$, $f(x)$ has the same sign as ℓ in an evironment of a;
3. if g is another function such that $\lim_{x\to b} g(x) = a$, but $g(x) \neq a$ in an environment of b, then $\lim_{x\to b} (f \circ g)(x) = \ell$; in particular, if $\ell > 0$:
   (a) $\lim_{x\to a} \log f(x) = \log \ell$;
   (b) $\lim_{x\to a} f(x)^\alpha = \ell^\alpha$, for any $\alpha \in \mathbb{R}$.

R An environment of a point $a \in \mathbb{R}$ is an interval of the form $(a-\delta, a+\delta)$, where $\delta > 0$. If a property holds "in an environment" it means that there exists some value $\delta > 0$ such that the property holds within the environment $(a-\delta, a+\delta)$.

■ **Example 5.7** Obviously $\lim_{x\to a} x = a$. But from this and property 2. above we can conclude that for all $n \in \mathbb{R}$

$$\lim_{x\to a} x^n = a^n.$$

A polynomial $P_n(x)$ is a linear combination of powers x^k, $k = 0, 1, \ldots, n$. Thus, applying properties 1. and 2. it follows that

$$\lim_{x\to a} P_n(x) = P_n(a). \tag{5.7}$$

Then, applying property 3.,

$$\lim_{x\to a} \frac{P_n(x)}{Q_m(x)} = \frac{P_n(a)}{Q_m(a)}, \tag{5.8}$$

provided a is not a root of $Q_m(x)$.

Thus, calculating limits of polynomials or rational functions is a trivial matter. ■

As for sequences, we have a Sandwich rule for functional limits:

Theorem 5.2.3 — Sandwich rule. If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = \ell$, and in an envionment of a it holds $g(x) \leqslant f(x) \leqslant h(x)$, then $\lim_{x\to a} f(x) = \ell$. (This rule is valid even if $a = \pm\infty$ or $\ell = \pm\infty$.)

A simple consequence of this sandwich rule is that

$$\lim_{x\to a} |f(x)| = 0 \qquad \Rightarrow \qquad \lim_{x\to a} f(x) = 0, \tag{5.9}$$

simply because $-|f(x)| \leqslant f(x) \leqslant |f(x)|$.

■ **Example 5.8** Let us apply the sandwich rule to calculate

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

We cannot simply apply the algebraic rules because $\sin\left(\frac{1}{x}\right)$ has no limit when $x \to 0$ (it oscillates more and more wildly between 0 and 1 as we approach 0). However,

$$-x^2 \leqslant x^2 \sin\left(\frac{1}{x}\right) \leqslant x^2$$

because for every $x \neq 0$ we have $-1 \leqslant \sin\left(\frac{1}{x}\right) \leqslant 1$. Since $\pm x^2 \to 0$ as $x \to 0$, by the sandwich rule

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

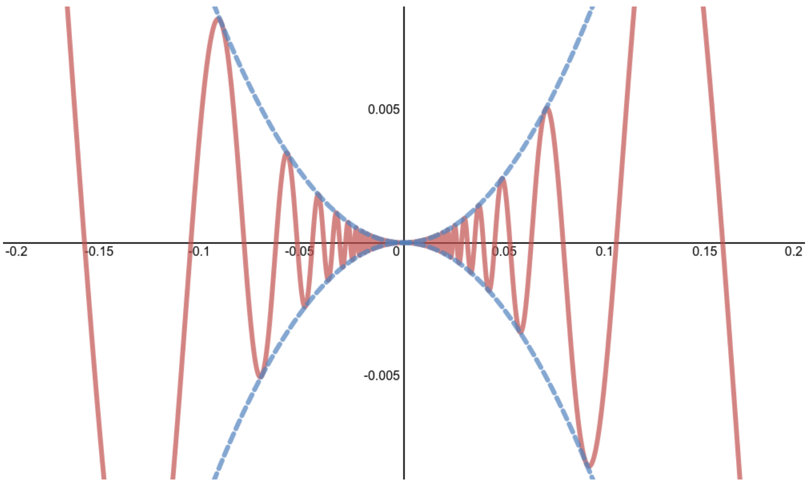

Figure 5.2: Plot of $f(x) = x^2 \sin\left(\frac{1}{x}\right)$. The dotted blue lines are plots of the two envelopes x^2 and $-x^2$.

Figure 5.2 shows a plot of $f(x)$ explaining intuitively what we have just proven analytically. ■

■ **Example 5.9** Let $a > 0$ and let us prove that

$$\lim_{x\to a} \sqrt{x} = \sqrt{a}.$$

First of all, notice that this is equivalent to proving that

$$\lim_{x\to a} \left(\sqrt{x} - \sqrt{a}\right) = 0 \qquad \Leftrightarrow \qquad \lim_{x\to a} \left|\sqrt{x} - \sqrt{a}\right| = 0$$

Now, to prove the latter we can write

$$0 \leqslant \left|\sqrt{x} - \sqrt{a}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leqslant \frac{|x-a|}{\sqrt{a}} \quad \xrightarrow[x\to a]{} \quad 0,$$

and the result follows from the sandwich rule. ■

Exercise 5.2 By a similar argument prove that for all $a > 0$

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}. \tag{5.10}$$

HINT: Use the identity $a - b = (a^n - b^n)/(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$. ■

■ **Example 5.10** From the equivalences (3.13), (3.14), it follows that for *any* sequence $\varepsilon_n \to 0$ we have

$$\lim_{n\to\infty} \sin\varepsilon_n = 0, \qquad \lim_{n\to\infty} \cos\varepsilon_n = 1, \qquad \lim_{n\to\infty} \frac{\sin\varepsilon_n}{\varepsilon_n} = 1, \qquad \lim_{n\to\infty} \frac{1-\cos\varepsilon_n}{\varepsilon_n^2} = \frac{1}{2}.$$

By the sequential definition of limit it follows that

$$\lim_{x\to 0} \sin x = 0, \qquad \lim_{x\to 0} \cos x = 1, \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}. \tag{5.11}$$

Using this limits and trigonometric identities we can easily prove that

$$\lim_{x\to a} \sin x = \sin a, \qquad \lim_{x\to a} \cos x = \cos a. \tag{5.12}$$

For instance, for the sine, setting $x = a + t$,

$$\begin{aligned}\lim_{x\to a} \sin x &= \lim_{t\to 0} \sin(a+t) = \lim_{t\to 0}(\sin a\cos t + \cos a \sin t) = \sin a \underbrace{\left(\lim_{t\to 0}\cos t\right)}_{=1} + \cos a \underbrace{\left(\lim_{t\to 0}\sin t\right)}_{=0} \\ &= \sin a,\end{aligned}$$

and similarly for the cosine. ■

5.3 Asymptotic comparison of functions

Example 5.10 makes it clear that, thanks to the sequential definition of limit, the notion of asymptotic comparison can be brought to the realm of functional limits. Hence we have:

Definition 5.3.1 — Asymptotic comparison. Let f and g be two real functions that either both diverge or both converge to 0 as $x \to a$ ($-\infty \leqslant a \leqslant \infty$). We say that f and g are **equivalent** when $x \to a$ (and denote it $f(x) \sim g(x)$ as $x \to a$) if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

Accordingly, given equations (3.13) and (3.14), we can state that

$$\log(1+x) \sim (e^x - 1) \sim \sin x \sim \tan x \sim x \qquad (x \to 0), \tag{5.13}$$

$$1 - \cos x \sim \frac{x^2}{2} \qquad (x \to 0). \tag{5.14}$$

(We will later re-derive this same relations in a more systematic and natural way.)

Problems

Problem 5.1 Calculate the following limits, simplifying the common factors that numerator and denominator may contain:

(i) $\lim_{x\to a} \frac{x^n - a^n}{x-a}$, $n \in \mathbb{N}$;

(ii) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$;

(iii) $\lim_{x\to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$;

(iv) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$;

(v) $\lim_{x\to 0} \frac{\frac{1}{(1-x)^3}-1}{x}$;

(vi) $\lim_{x\to 1} \left(\frac{1}{\sqrt{x}-1} - \frac{2}{x-1}\right)$.

Problem 5.2 Calculate the following limits:

(i) $\lim_{x\to 0} \frac{(\sin 2x^3)^2}{x^6}$;

(ii) $\lim_{x\to 0} \frac{\tan x^2 + 2x}{x+x^2}$;

(iii) $\lim_{x\to 0} \frac{\sin(x+a)-\sin a}{x}$;

(iv) $\lim_{x\to 0} (1+x)^{1/x}$;

(v) $\lim_{x\to 0} \frac{\log(1-2x)}{\sin x}$;

(vi) $\lim_{x\to 0} (1+\sin x)^{2/x}$;

(vii) $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x-\sin x}$;

(viii) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$;

(ix) $\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}}$;

(x) $\lim_{x\to 0} (\cos x)^{1/x^2}$;

(xi) $\lim_{x\to \pi} \frac{1-\sin(x/2)}{(x-\pi)^2}$;

(xii) $\lim_{x\to 0} \frac{a^x - b^x}{x}$.

Problem 5.3 Calculate the following limits:

(i) $\lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}$;

(ii) $\lim_{x\to\infty} \frac{x+\sin x^3}{5x+6}$;

(iii) $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$;

(iv) $\lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right)$;

(v) $\lim_{x\to\pm\infty} \frac{e^x}{e^x-1}$;

(vi) $\lim_{x\to\pm\infty} \frac{x-2}{\sqrt{4x^2+1}}$;

(vii) $\lim_{x\to\pm\infty} \tanh x$;

(viii) $\lim_{x\to\pm\infty} \frac{e^x}{\sinh x}$;

(ix) $\lim_{x\to\pm\infty} \left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}$.

Problem 5.4 Calculate the one-sided limits:

(i) $\lim_{x\to 0^\pm} \left(\frac{1}{x}\right)^{\lfloor x\rfloor}$;

(ii) $\lim_{x\to 0^\pm} e^{1/x}$;

(iii) $\lim_{x\to 0^\pm} \frac{1-e^{1/x}}{1+e^{1/x}}$.