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Calculus I

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Unit 6. Continuity

6. Continuity

6.1 Definition, properties, and continuity of elementary functions

Those functions whose limit at a point a of their domain coincides with the value of that function at that point play a very special role in calculus. They mainly coincide with those functions whose graph "can be plotted without lifting the pen from the paper" —which is the intuitive notion of a continuous function.[1] The formal definition of continuity is the following:

Definition 6.1.1 — Continuity. A real function f is said to be **continuous** at a point a of its domain if

$$\lim_{x \to a} f(x) = f(a). \tag{6.1}$$

Definition 6.1.2 — Continuity in intervals. f is said to be continuous in

(a,b) if it is continuous at every point $x \in (a,b)$;
$[a,b)$ if it is continuous in (a,b) and $\lim_{x \to a^+} f(x) = f(a)$;
$(a,b]$ if it is continuous in (a,b) and $\lim_{x \to b^-} f(x) = f(b)$;
$[a,b]$ if it is continuous in $[a,b)$ and in $(a,b]$.

■ **Example 6.1** Examples 5.7 and 5.10 prove that:

(a) polynomials $P_n(x)$ are continuous in all $\mathbb{R}$;
(b) rational functions $P_n(x)/Q_m(x)$ are continuous in all $\mathbb{R}$ except at the roots of Q_m;
(c) $\sin x$ and $\cos x$ are continuous in all $\mathbb{R}$;
(d) $\tan x$ is continuous except at the zeroes of $\cos x$;
(e) $\cot x$ is continuous except at the zeroes of $\sin x$.

■

[1] We say 'mainly' because there are very weird functions, which one would intuitively not refer to them as continuous, and nevertheless they are continuous in some subsets. But we shall not be concerned with these functions in this course. We will rather focus on practical, "sensible" functions.

■ **Example 6.2** Let us consider the exponential function. At any point $a \in \mathbb{R}$ we can write

$$e^x = e^{a+(x-a)} = e^a e^{x-a} = e^a + e^a(e^{x-a} - 1).$$

Since $e^{x-a} - 1 \sim x - a$ as $x \to a$,

$$\lim_{x\to a} e^x = e^a + e^a \lim_{x\to a}(x-a) = e^a.$$

Therefore the exponential function is continuous in $\mathbb{R}$. ■

The algebraic properties of functional limits yield the following algebraic properties for continuous functions:

Proposition 6.1.1

(i) If f and g are continuous at a, then so are $f+g$ and fg. If on top of that $g(a) \neq 0$, then f/g is also continuous at a.

(ii) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

(iii) If an invertible function f is continuous at a, then f^{-1} is continuous at $f(a)$.

■ **Example 6.3** As a consequence of (iii) in the previous Proposition, $\log x$, $\arcsin x$, $\arccos x$, $\arctan x$, $\operatorname{arccot} x$ are continuous functions in their domains. ■

■ **Example 6.4** Function $f(x) = x^\alpha$, for $\alpha \in \mathbb{R}$, is continuous in $(0,\infty)$. The reason is that f can be written

$$f(x) = x^\alpha = e^{\alpha \log x},$$

i.e., as a composition of continuous functions.

If $\alpha > 0$ and we define $f(0) = 0$, then it is also continuous at $x = 0$ because

$$\lim_{x\to 0^+} e^{\alpha \log x} = \lim_{t\to -\infty} e^t = 0$$

(we have made the change of variable $t = \alpha \log x$).

If $\alpha = 0$ then $f(x) = 1$ in $(0,\infty)$, so it is continuous also at $x = 0$ if we define $f(0) = 1$. ■

6.2 Discontinuities

Discontinuities are points where a function is not continuous. There are several reasons why a function may not be continuous at a point, and some of them bear a specific name.

A function like $f(x) = \dfrac{\sin x}{x}$ is continuous in all $\mathbb{R}$ except $x = 0$, because the denominator vanishes at that point. However, the function has a well defined limit at that point (see equation (5.11)),

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

So we can re-define the function to be

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1 & x = 0, \end{cases}$$

and now it is continuous everywhere in $\mathbb{R}$. One such discontinuity is called an **avoidable discontinuity** because it can be "avoided" by properly defining the function.

The case of the Heaviside step function

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0, \end{cases}$$

typifies a stronger case of discontinuity, which cannot be avoided. The function is continuous in $\mathbb{R} - \{0\}$ (because it is a constant for $x < 0$ and for $x > 0$), but at $x = 0$ the left-handed limit is 0 whereas the right-handed limit is 1. So the limit when $x \to 0$ does not exist because, although the two one-sided limits exist, they are different. This is a **jump discontinuity** because the graph of the function "jumps" at that point.

R Note that, given the definition of continuity in intervals that we have made, although $H(x)$ is discontinuous at $x = 0$, it is continuous e.g. in $[0, 1]$ (but it not continuous in $[-1, 0]$).

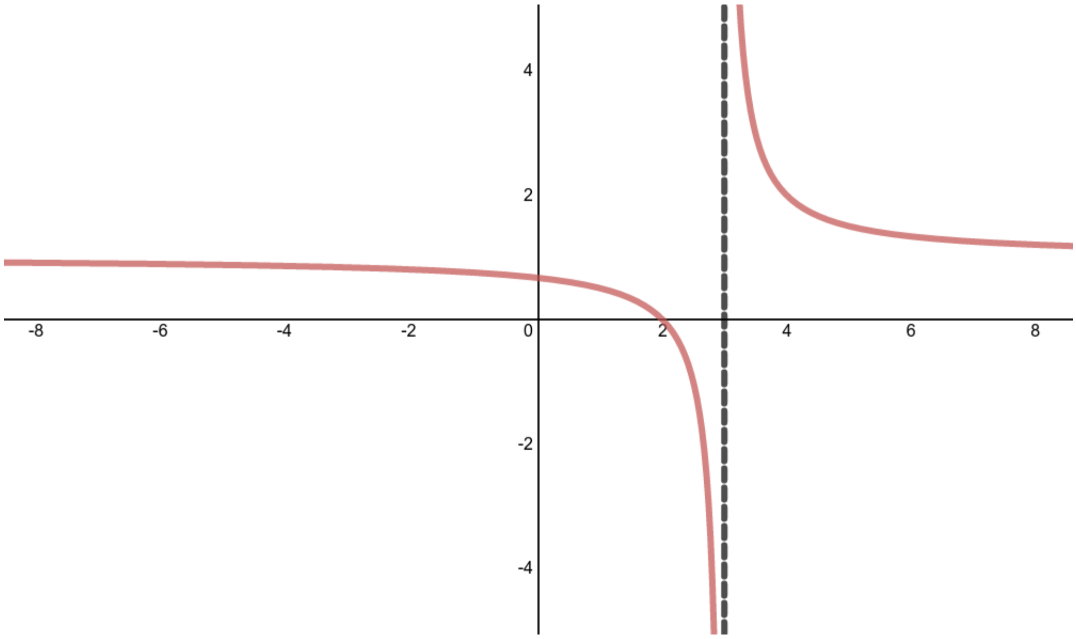

Figure 6.1: Illustration of an asymptote for the function $f(x) = \frac{x-2}{x-3}$.

In some cases the function is not continuous because the one or both of the two one-sided limits is $\pm\infty$. Such is the case of $1/x$ or $\log x$. We say that the function has a **singularity** at that point. We also call it an **asymptote** (see Figure 6.1).

Finally, a function can be discontinuous simply because it has no limit at a point. For instance, $\sin \frac{1}{x}$ is continuous in $\mathbb{R} - \{0\}$ because the limit when $x \to 0$ does not exist.

Exercise 6.1 Which kind of discontinuity has the function $f(x) = x \sin \frac{1}{x}$ at $x = 0$? ■

6.3 Continuous functions in closed intervals

Continuity in a closed interval is a very restricting property. As a consequence, knowing that a function is continuous in a closed interval provides us very relevant information about the function. This information is captured in a series of theorems, the most important of which is due to Bolzano.

Theorem 6.3.1 — Bolzano's theorem. If f is a continuous function in $[a, b]$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

This theorem is the formal expression of the well-known fact that we cannot draw a continuous curve from a point above the X axis to another point below without crossing the X axis.

This theorem has an important corollary that basically expresses the same idea.

Corollary 6.3.2 — Intermediate values theorem. If f is a continuous function in $[a,b]$ and $\min\{f(a),f(b)\} < z < \max\{f(a),f(b)\}$, then there exists $c \in (a,b)$ such that $f(c) = z$.

We reword this result by stating that a continuous function in a closed interval $[a,b]$ takes all intermediate values between $f(a)$ and $f(b)$.

Proof. The proof is as simple as defining the function $g(x) = f(x) - z$, which being the sum of two continuous functions is continuous itself in $[a,b]$. But either $f(a) < z < f(b)$ (i.e., $f(a) - z < 0 < f(b) - z$) or $f(b) < z < f(a)$ (i.e., $f(b) - z < 0 < f(a) - z$); in any case, $g(a)g(b) < 0$. Then Bolzano's theorem implies that there exists $c \in (a,b)$ such that $g(c) = f(c) - z = 0$. ■

■ **Example 6.5** We shall prove, applying the intermediate value theorem, that the equation $xe^x = 1$ has a solution $x^* > 0$.

Let us define the function $f(x) = xe^x$, which is continuous in $\mathbb{R}$. Now, $f(0) = 0$ and $f(1) = e = 2.71828\ldots$ Then $f(0) < 1 < f(1)$, so 1 is an intermediate value of those f takes in the interval $[0,1]$. According to the intermediate values theorem there exists $0 < x^* < 1$ such that $f(x^*) = 1$, and that is the solution we are looking for. ■

The last important result of this sort is the following theorem:

Theorem 6.3.3 If f is a continuous function in the interval $[a,b]$ then there exists $x_m, x_M \in [a,b]$ such that $f(x_m) \leqslant f(x) \leqslant f(x_M)$ for any $x \in [a,b]$.

In other words, a continuous function in a closed interval reaches its maximum and minimum values within the interval (in particular, it is bounded).

The requirement of f to be continuous is well illustrated by the function $f(x) = 1/x$ in $[-1,1]$. It is not even bounded because is not continuous in the interval (as a matter of fact, it is not even defined at $x = 0$).

The requirement of the interval to be closed is illustrated, for instance, by the function $f(x) = x^2$ in $[0,1)$. Although the function is continuous in the whole interval it does not reach the maximum within it (the supreme of the values of f in that interval is $f(1) = 1$, but it is clearly reached outside the interval).

Problems

Problem 6.1

(a) Prove that if f is continuous then so is $|f|$. Show that the reciprocal is false by finding a counterexample.

(b) What can be said about a function that is continuous but all the values it takes are in $\mathbb{Q}$?

Problem 6.2

(a) Let $f:[0,1]\to[0,1]$ be a continuous, surjective function. Prove that there exists $c\in[0,1]$ such that $f(c)=c$.

(b) Let f be a continuous function in $[a,b]$ and let $x_1,\dots,x_n\in[a,b]$. Prove that there exists $c\in[a,b]$ such that $f(c)=\frac{1}{n}\sum_{k=1}^{n}f(x_k)$.

Problem 6.3 Consider the function

$$f(x)=\frac{1}{\lambda x^2-2\lambda x+1}.$$

Determine for which values $\lambda\in\mathbb{R}$ the function is continuous in (a) $\mathbb{R}$, or (b) $[0,1]$.

Problem 6.4 Study the continuity of the following functions:

(i) $f(x)=\dfrac{e^{-5x}+\cos x}{x^2-8x+12}$;

(ii) $f(x)=e^{3/x}+x^3-9$;

(iii) $f(x)=x^3\tan(3x+2)$;

(iv) $f(x)=\sqrt{x^2-5x+6}$;

(v) $f(x)=(\arcsin x)^3$;

(vi) $f(x)=(x-5)\log(8x-3)$;

(vii) $f(x)=x-\lfloor x\rfloor$;

(viii) $f(x)=\begin{cases}x^2\sin(1/x), & x\neq 0,\\ 0, & x=0;\end{cases}$

(ix) $f(x)=\begin{cases}\dfrac{\tan x}{\sqrt{x}}, & x>0,\\ 0, & x=0,\\ e^{1/x}, & x<0;\end{cases}$

(x) $f(x)=\begin{cases}x, & x\in\mathbb{Q},\\ -x, & x\notin\mathbb{Q};\end{cases}$

(xi) $f(x)=\begin{cases}\sin(\pi x), & x<-1,\\ |x|-x, & -1\leqslant x<1,\\ (x-1)^3, & x\geqslant 1;\end{cases}$

(xii) $f(x)=\begin{cases}(x+1)^2, & x\leqslant -1,\\ \operatorname{sgn}x+1, & -1<x<1,\\ 2x, & x\geqslant 1;\end{cases}$

(xiii) $f(x)=\begin{cases}x^2, & x\leqslant -2,\\ |x^2-1|, & -2<x<2,\\ 4x-5, & x\geqslant 2;\end{cases}$

(xiv) $f(x)=\begin{cases}(x-1)^2, & x>1,\\ x-\lfloor x\rfloor, & -1\leqslant x\leqslant 1,\\ x+1, & x<-1.\end{cases}$

Problem 6.5 Which of these equations have at least one solution in the specified set?:

(i) $x^2-18x+2=0$, in $[-1,1]$;

(ii) $x-\sin x=1$, in $\mathbb{R}$;

(iii) $e^x+1=0$, in $\mathbb{R}$;

(iv) $\cos x+2=0$, in $\mathbb{R}$;

(v) $f(x)=0$, in $[-2,2]$, where f is given by
$$f(x)=\begin{cases}x^2+2, & -2\leqslant x<0,\\ -(x^2+2), & 0\leqslant x\leqslant 2;\end{cases}$$

(vi) $\frac{1}{4}x^3-\sin(\pi x)+3=\frac{7}{3}$, in $[-2,2]$;

(vii) $|\sin x|=\sin x+3$, in $\mathbb{R}$.

Problem 6.6 Prove that any polynomial of odd degree has at least one real root.