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Calculus I

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Unit 7. Derivatives

7. Derivatives

7.1 Concept and definition

Derivates are introduced to characterise the *rate of variation* of a function with a number. The rate of variation measures how much the function $f(x)$ increases (positive) or decreases (negative) per unit of variation of the variable x. Thus, within the interval $[a,x]$ this rate will be

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

Figure 7.1 illustrates that the narrower the interval $[a,x]$ where the variation is measured the more accurate the estimated rate. Ideally, the measure would be perfect if this interval were infinitely narrow. This is the notion of *derivative* and the motivation of its definition:

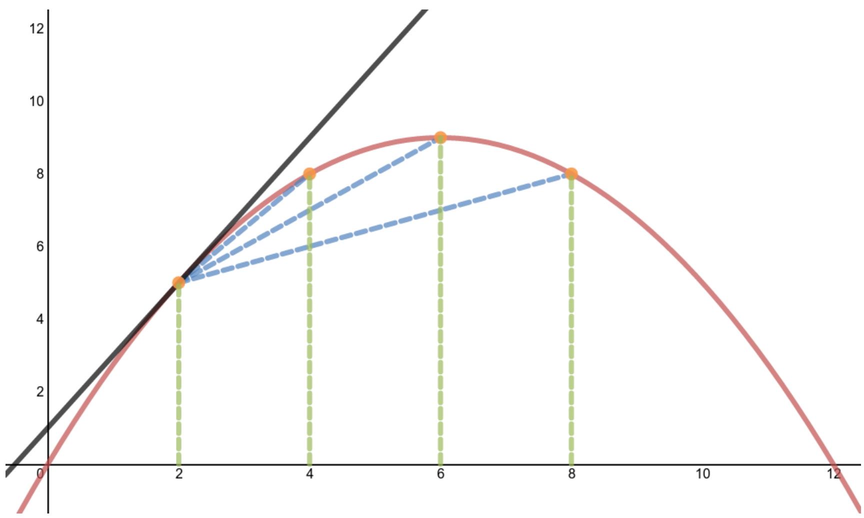

Figure 7.1: The rate of variation of $f(x)$ as obtained for different intervals.

Definition 7.1.1 — Derivative. The **derivative** of the function f at the point a of its domain is defined as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \tag{7.1}$$

provided the limit exists. (When it does, we say that the function is *differentiable* at a.)

Alternatively, introducing the change of variable $x = a + h$, the limit (7.1) can be obtained as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Figure 7.1 also shows that $f'(a)$ —the rate of variation of $f(x)$ at $x = a$— coincides with the *slope* of the straight line *tangent* to the graph of $f(x)$ at the point $\big(a, f(a)\big)$ —which is an important geometric characterisation of the derivative concept.

R Often you will see the derivative denoted as

$$f'(a) = \frac{df}{dx}(a).$$

This is Leibniz's notation —a bit more mnemotechnical because it reminds that the derivative is, after all, a rate of change of f.

■ **Example 7.1** Consider the function $f(x) = x^2$. Its derivative at any point x would be, according to the defintion,

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

Therefore $f'(x) = 2x$. ■

Exercise 7.1 Using Newton's binomial formula prove that the derivative of $f(x) = x^n$, with $n \in \mathbb{N}$ arbitrary, at any point $x \in \mathbb{R}$ is $f'(x) = nx^{n-1}$. (Note that this formula holds even if $n = 0$, for which $f(x) = 1$.) ■

■ **Example 7.2** Let $f(x) = \sin x$ and $g(x) = \cos x$. By definition

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}. \end{aligned}$$

But

$$\lim_{h \to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h \to 0} \frac{\cos h - 1}{h} = -\lim_{h \to 0} h \frac{1 - \cos h}{h^2} = -0 \cdot \frac{1}{2} = 0,$$

hence $f'(x) = \cos x$.

Similarly

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} = -\sin x. \end{aligned}$$

Thus we have the result $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$. ■

■ **Example 7.3** Let $f(x) = e^x$ and compute

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

■

We say that f is differentiable in the interval (a,b) if it is differentiable at every point of the interval.

The function f', defined as

$$\begin{aligned} f' : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f'(x), \end{aligned} \tag{7.2}$$

where A is the set of points where f is differentiable, is called the **derivative function** of f (or simply the *derivative* of f).

Likewise, we can introduce higher order derivatives. For instance, f'' is the *second derivative* of f, i.e., the derivative function of f'. Or f''' is the *third derivative* of f, i.e., the derivative function of f''. And so on. (Beyond the third derivative it is customary to denote higher order derivatives as $f^{(n)}$, the nth derivative of f.)

The following theorem emphasises that differentiability is a more restrictive property than continuity.

Theorem 7.1.1 If f is differentiable at a it is also continuous at a.

Proof. It is very simple. All we have to prove is that

$$\lim_{x\to a} f(x) = f(a) \qquad \Leftrightarrow \qquad \lim_{x\to a}[f(x) - f(a)] = 0.$$

For that we just need to multiply and divide by $x - a$, and apply the algebraic properties of limits:

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) = f'(a) \cdot 0 = 0,$$

where we have used that the first limit exists (the hypothesis of the theorem) and it is the derivative of f at $x = a$. ■

An obvious consequence of this theorem is that discontinuous functions are not differentiable at the discontinuities.

■ **Example 7.4** Function $f(x) = |x|$ is continuous in $\mathbb{R}$, however, $f'(0)$ does not exist. The reason is that

$$\lim_{x\to 0^+} \frac{|x| - 0}{x - 0} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

because $|x| = x$ for $x \geqslant 0$. However

$$\lim_{x\to 0^-} \frac{|x| - 0}{x - 0} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

because $|x| = -x$ for $x < 0$. Therefore the limit defining $f'(0)$ does not exists because the left-handed and right-handed limits are different. ■

■ **Example 7.5** Function

$$f(x)=\begin{cases} x\sin(1/x), & x\neq 0,\\ 0, & x=0\end{cases}$$

is continuous in $\mathbb{R}$, however, $f'(0)$ does not exist:

$$\lim_{x\to 0}\frac{x\sin(1/x)-0}{x-0}=\lim_{x\to 0}\sin(1/x).$$

On the contrary, function

$$g(x)=\begin{cases} x^2\sin(1/x), & x\neq 0,\\ 0, & x=0\end{cases}$$

is differentiable at $x=0$ (in fact everywhere, as we will see later) and

$$f'(0)=\lim_{x\to 0}\frac{x^2\sin(1/x)-0}{x-0}=\lim_{x\to 0}x\sin(1/x)=0.$$

■

7.2 Algebraic properties

The fact that derivatives are defined as limits leads to the following algebraic properties:

Proposition 7.2.1 Let f and g be two differentiable functions (in an appropriate set). Then:

(i) $(\lambda f+\mu g)'=\lambda f'+\mu g'$, where $\lambda,\mu\in\mathbb{R}$; *(linearity)*

(ii) $(fg)'=f'g+fg'$; *(Leibniz's rule)*

(iii) $(f\circ g)'=(f'\circ g)g'$; *(chain rule)*

(iv) $\left(\dfrac{f}{g}\right)'=\dfrac{f'g-fg'}{g^2}$, provided $g\neq 0$; *(quotient rule)*

(v) $(f^{-1})'=\dfrac{1}{f'\circ f^{-1}}$; *(inverse rule)*

Proof. Except for some technicalities —which we will omit here—, the proof of these rules is just an application of the algebraic properties of limits.

(i) From the linearity of limits,

$$\begin{aligned}(\lambda f+\mu g)'(a)&=\lim_{x\to a}\frac{\lambda f(x)+\mu g(x)-\lambda f(a)+\mu g(a)}{x-a}\\ &=\lambda\lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\mu\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\\ &=\lambda f'(a)+\mu g'(a).\end{aligned}$$

(ii) Now we need to add and substract $f(a)g(x)$:

$$\begin{aligned}(fg)'(a)&=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}\\ &=\lim_{x\to a}g(x)\frac{f(x)-f(a)}{x-a}+f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=g(a)f'(a)+f(a)g'(a),\end{aligned}$$

where we have used that g is continuous because it is differentiable.

(iii) Here we need to multiply and divide by $g(x)-g(a)$:

$$(f\circ g)'(a)=\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{x-a}=\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{g(x)-g(a)}\frac{g(x)-g(a)}{x-a}$$
$$=\lim_{y\to g(a)}\frac{f(y)-f\big(g(a)\big)}{y-g(a)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=f'\big(g(a)\big)g'(a)=(f'\circ g)(a)g'(a),$$

where we have changed the variable $y=g(x)$, so that $y\to g(a)$ as $x\to a$ because g is continuous.

(iv) First we need to prove $(x^{-1})'=-x^{-2}$ for any $x\neq 0$. We do that using the definition:

$$\left(\frac{1}{x}\right)'=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}=\lim_{h\to 0}\frac{-h}{hx(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}.$$

Now we write the quotien as a product and apply the product rule:

$$\left(\frac{f}{g}\right)'=\left(f\cdot\frac{1}{g}\right)'=f'\left(\frac{1}{g}\right)+f\left(\frac{1}{g}\right)'.$$

But $1/g=h\circ g$, where $h(x)=x^{-1}$, so we can apply the chain rule and obtain

$$\left(\frac{1}{g}\right)'=-\frac{g'}{g^2}.$$

Thus, finally,

$$\left(\frac{f}{g}\right)'=\frac{f'}{g}-\frac{fg'}{g^2}=\frac{f'g-fg'}{g^2}.$$

(v) The equation that defines the inverse is $(f\circ f^{-1})(x)=x$. If we differentiate this equation, we get

$$\left(f\circ f^{-1}\right)'(x)=1.$$

Applying the chain rule we obtain

$$\left(f'\circ f\right)(x)\left(f^{-1}\right)'(x)=1,$$

from which we arrive at the result

$$\left(f^{-1}\right)'(x)=\frac{1}{\left(f'\circ f\right)(x)}.$$

■

The following examples illustrate how these rules can be applied to obtain new derivatives:

■ **Example 7.6** If $f(x)=e^x$ then $f^{-1}(x)=\log x$. Therefore

$$(\log x)'=\frac{1}{e^{\log x}}=\frac{1}{x}.$$

Logarithms can have a different base, say $a>0$. They are denoted $\log_a x$ and form the inverse function of a^x. Now $a^x=e^{x\log a}$, so by the chain rule

$$(a^x)'=\left(e^{x\log a}\right)'=\left(e^{x\log a}\right)\log a=a^x\log a.$$

Therefore

$$(\log_a x)'=\frac{1}{a^{\log_a x}\log a}=\frac{1}{x\log a}.$$

■

■ **Example 7.7** Function $f(x) = x^\alpha$, with $\alpha \in \mathbb{R}$, can be written as $f(x) = e^{\alpha \log x}$. Thus, applying the chain rule,

$$(x^\alpha)' = e^{\alpha \log x} \frac{\alpha}{x} = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha - 1}.$$

■

■ **Example 7.8** If $f(x) = \sin x$ in $[-\pi/2, \pi/2]$ then $f^{-1}(x) = \arcsin x$. Thus,

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$

But $\cos x = \sqrt{1 - \sin^2 x}$ in $[-\pi/2, \pi/2]$, so

$$(\arcsin x)' = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}},$$

because $\sin(\arcsin x) = x$. ■

Exercise 7.2 Calculate the derivative of the functions $\tan x$ and $\arctan x$. ■

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\sin x$	$\cos x$	$\arctan x$	$\frac{1}{1+x^2}$
x^α	$\alpha x^{\alpha-1}$	$\cos x$	$-\sin x$	$\operatorname{arccot} x$	$\frac{-1}{1+x^2}$
e^x	e^x	$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\sinh x$	$\cosh x$
a^x	$a^x \log a$	$\cot x$	$\frac{-1}{\sin^2 x} = -1 - \cot^2 x$	$\cosh x$	$\sinh x$
$\log x$	$\frac{1}{x}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\tanh x$	$\frac{1}{\cosh^2 x} = 1 - \tanh^2 x$
$\log_a x$	$\frac{1}{x \log a}$	$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\coth x$	$\frac{-1}{\sinh^2 x} = 1 - \coth^2 x$

Table 7.1: Derivatives of most elementary functions. Here $c, \alpha \in \mathbb{R}$, $a > 0$.

7.3 Derivatives and local behaviour

We will see here a set of results related to the local behaviour of a function (i.e., the behaviour within intervals). To begin with, we need to define local maxima and minima.

We say that a function f has a **local maximum** at a point a of its domain, if there is some interval $(a - \delta, a + \delta)$ such that $f(x) \leqslant f(a)$ for all $x \in (a - \delta, a + \delta)$.

We say that a function f has a **local minimum** at a point a of its domain, if there is some interval $(a - \delta, a + \delta)$ such that $f(x) \geqslant f(a)$ for all $x \in (a - \delta, a + \delta)$.

Local maxima and minima are collectively called **local extrema.** If local extrema remain extrema for all x in the domain of f, they are **absolute extrema.**

Theorem 7.3.1 — Derivatives at local extrema. If f has a local extremum at a point a where it is differentiable then $f'(a) = 0$.

Proof. Consider the case of a maximum (the proof for a minimum is analogous). By the definition of local maximum

$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \geqslant 0$$

because $f(x) \leqslant f(a)$ near a, so $f(x) - f(a) \leqslant 0$, but on the left of a we have $x - a \leqslant 0$. On the other hand,

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \leqslant 0$$

because the numerator is again $f(x) - f(a) \leqslant 0$, but on the right of a we have $x - a \leqslant 0$. Since the derivative exists both limits must coincide, so the only possibility is that both are 0. Hence $f'(a) = 0$. ■

■ **Example 7.9** Consider the function $f(x) = |x(1-x)|$. We know that $x(1-x) \geqslant 0$ if $0 \leqslant x \leqslant 1$, and $x(1-x) < 0$ if $x < 0$ or $x > 1$. Then we can rewrite

$$f(x) = \begin{cases} x(1-x), & 0 \leqslant x \leqslant 1, \\ x(x-1), & x < 0 \text{ or } x > 1. \end{cases}$$

Let us compute the derivative,

$$f'(x) = \begin{cases} 1-2x, & 0 < x < 1, \\ 2x-1, & x < 0 \text{ or } x > 1. \end{cases}$$

The derivative at $x = 0$ and $x = 1$ does not exists because, being $f(0) = 0$ and $f(x) = x(x-1)$ for $x < 0$,

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x(x-1)}{x} = \lim_{x\to 0^-} (x-1) = -1.$$

However, since $f(x) = x(1-x)$ for $x > 0$,

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x(1-x)}{x} = \lim_{x\to 0^+} (1-x) = 1.$$

Since both one-sided limits are different the limit does not exist. For $x = 1$ the argument is similar.

Now to find the local extrema we need to look for the solutions of $f'(x) = 0$. This equation boils down to $2x = 1$, whose solution is $x = \frac{1}{2}$.

Figure 7.2 presents a plot of $f(x)$. One can clearly see that $x = \frac{1}{2}$ is indeed a local maximum —albeit not absolute, because there are points where $f(x) > f(1/2)$—; however, we can also see that $x = 0$ and $x = 1$ are local minima, but they are not contained in the equation $f'(x) = 0$. (Incidentally, these minima are both absolute.)

There is no contradiction with the theorem though, because, as we have just seen, the function is not differentiable at those points —a premise of the theorem.

This example brings about the point that, when looking for extrema, we need to check not only the solutions of $f'(x) = 0$, but also the points where $f'(x)$ does not exist. ■

> **R** Notice also that $f'(c) = 0$ does not imply that c is an extremum. For instance take $f(x) = x^3$. Clearly $f'(0) = 0$, however there is no extremum at $x = 0$ because $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$.

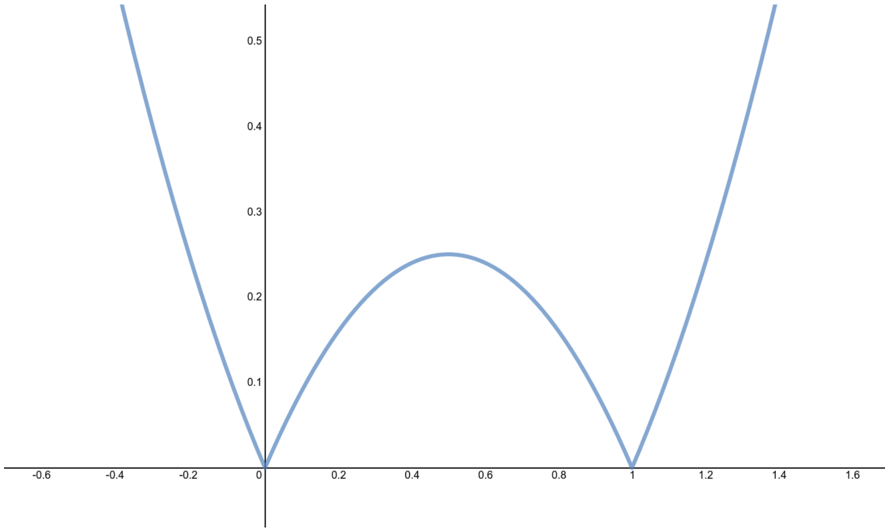

Figure 7.2: Plot of the function $f(x) = |x(1-x)|$.

Theorem 7.3.2 — Rolle's theorem. Let f be continuous in $[a,b]$, differentiable in (a,b), and such that $f(a) = f(b)$; then there exists $c \in (a,b)$ where $f'(c) = 0$.

Proof. The proof is very simple. Every continuous function in a closed interval reaches its absolute maximum and minimum within that interval. There are three possibilities:

(a) Both extrema are in (a,b). In that case f will be differentiable at both of them, c_{min} and c_{max}. According to Theorem 7.3.1 $f'(c_{\text{min}}) = f'(c_{\text{max}}) = 0$.

(b) One extremum is at $x = a$ or at $x = b$ and the other one is at $c \in (a,b)$. Then $f'(c) = 0$.

(c) One extremum is at $x = a$ and the other one at $x = b$. Then the function must be a constant because $f(a) = f(b)$, and its derivative will be $f'(x) = 0$ everywhere in (a,b).

In any of the three cases we see that $f'(c) = 0$ in at least one point of (a,b). ■

Theorem 7.3.3 — Mean value theorem. Let f be continuous in $[a,b]$ and differentiable in (a,b); then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

Exercise 7.3 Prove the mean value theorem by applying Rolle's theorem to the function

$$g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x-a) - f(a).$$

(First check that g satifies the hypotheses of the theorem.) ■

There are practical consequences of the mean value theorem, which can be summarised in this corollary:

Corollary 7.3.4 With the hypothesis of the mean value theorem:

(i) If $f'(x) = 0$ for all $x \in (a,b)$ then f is constant in (a,b).

(ii) If $f'(x) = g'(x)$ for all $x \in (a,b)$ then $f(x) = g(x) + k$ in (a,b), with $k \in \mathbb{R}$ a constant.

(iii) If $f'(x) > 0$ for all $x \in (a,b)$ then f is strictly increasing in (a,b).

(iv) If $f'(x) < 0$ for all $x \in (a,b)$ then f is strictly decreasing in (a,b).

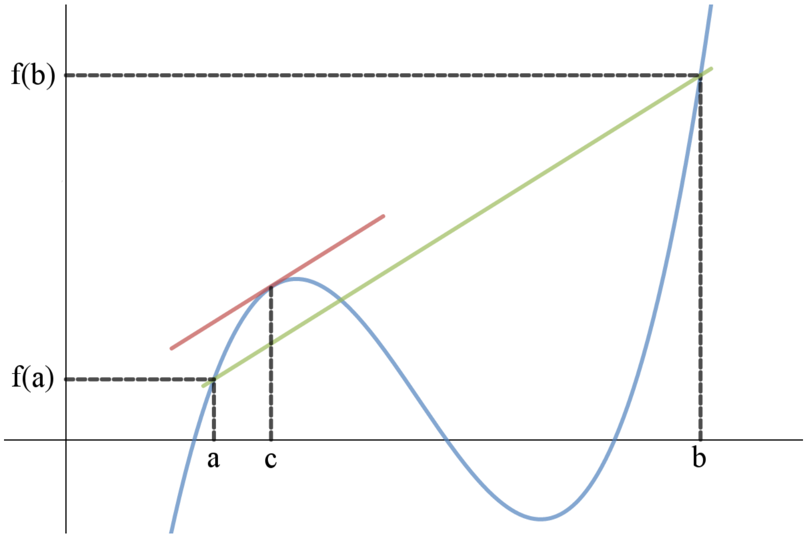

Figure 7.3: Geometric interpretation of the mean value theorem. The green straight line joining the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ is defined by the equation $y = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(a)$. The red parallel line proves that there is at least one point in the curve, $\big(c, f(c)\big)$, where the tangent has the same slope as (is parallel to) the straight line. This is the statement of the theorem.

Proof. Simply take $a < x < y < b$ arbitrary and apply the mean value theorem:

$$f(y) - f(x) = f'(c)(y-x),$$

where $x < c < y$.

If we assume $f'(c) = 0$ for any $c \in (a,b)$, then $f(y) = f(x)$. Since this is valid for any pair of points x, y in (a,b) this proves (i).

If we assume $f'(c) > 0$ for any $c \in (a,b)$, then $f(y) > f(x)$ whenever $y > x$. Since this is valid for any pair of points x, y in (a,b) this proves (iii).

If we assume $f'(c) < 0$ for any $c \in (a,b)$, then $f(y) < f(x)$ whenever $y > x$. Since this is valid for any pair of points x, y in (a,b) this proves (iv).

As for (ii), it is just a consequence of applying (i) to the function $f - g$. ■

These resuls are useful in identifying the nature of extrema, as this example illustrates:

■ **Example 7.10** Find the absolute extrema of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ in the interval $[-8, 1]$.

There are four steps to solve a problem like this:

(1) Find the set where $f'(x)$ exists, and solve the equation $f'(x) = 0$ within that set.
(2) Take all solutions of $f'(x) = 0$ along with the points where $f'(x)$ does not exist.
(3) Check whether any of those point is a local extremum by checking the sign of f' on their left and on their right.
(4) Compare the value of $f(x)$ in all those points as well as the values at the extremes of the interval. Select the largest and the smallest and identify the absolute extrema.

In the case we are dealing with here

$$f'(x) = \frac{10}{3}(x^{2/3} + x^{-1/3}) = \frac{10}{3}(x+1)x^{-1/3}.$$

This function is well defined for all $x \neq 0$. At $x = 0$ the derivative does not exists because the limit

$$\lim_{x\to 0} \frac{2x^{5/3} + 5x^{2/3}}{x} = \lim_{x\to 0}\left(2x^{2/3} + 5x^{-1/3}\right)$$

diverges.

Now, the solution of $f(x) = 0$ is $x = -1$, and $f'(x) > 0$ for $x < -1$ (notice that $x^{-1/3} < 0$ whenever $x < 0$), but $f'(x) < 0$ for $-1 < x < 0$. The function thus increases on the left of $x = -1$ and decreases on the right, therefore there is a *local maximum* at $x = -1$.

As for $x = 0$, $f'(x) < 0$ for $-1 < x < 0$, but $f'(x) > 0$ for $x > 0$. Thus there is a *local minimum* at $x = 0$.

That is all for local extrema. Concerning absolute extrema we need to compute

$$f(-1) = 3, \qquad f(0) = 0, \qquad f(-8) = -44, \qquad f(1) = 7.$$

So the absolute maximum is at $x = 1$ (the rightmost extreme of the interval) and the absolute minimum is at $x = -8$ (the leftmost extreme of the interval).

Figure 7.4 illustrates what we have just found. ■

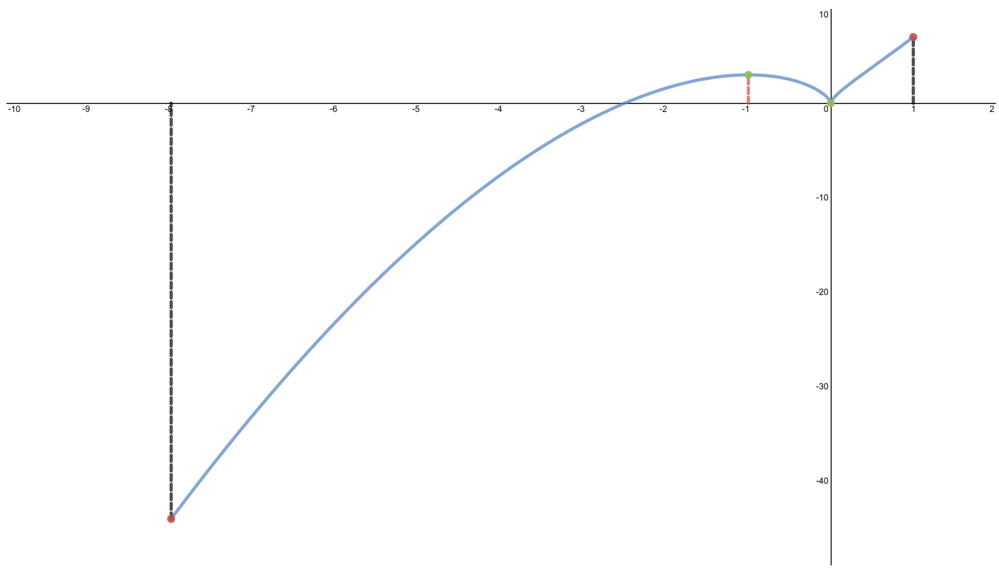

Figure 7.4: Plot of the function $f(x) = 2x^{5/3} + 5x^{2/3}$.

Theorem 7.3.5 — Cauchy's mean value theorem. Let f and g be both continuous in $[a,b]$ and differentiable in (a,b); then there exists $c \in (a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c). \tag{7.3}$$

Exercise 7.4 Prove Cauchy's mean value theorem by applying Rolle's theorem to the function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

(First check that h satifies the hypotheses of the theorem.) ■

Cauchy's mean value theorem is the basis for the proof of an important result in the calculations of limits of indeterminacies of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The theorem can also be applied to sequences—for which it is often an alternative to Stolz's theorem. The theorem (or rule, as it is customary referred to) is named after the 17th-century French mathematician Guillaume de l'Hôpital (1661–1704), although it was first introduced to him in 1694 by the Swiss mathematician Johann Bernoulli (1667–1748).

Theorem 7.3.6 — l'Hôpital's rule. Let f and g be two functions such that $g'(x) \neq 0$ in an environment of a (perhaps excluding a itself) and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \ell.$$

If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$, are both 0 or $\pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \ell.$$

R L'Hôpital rule remains valid even if $a = \pm\infty$ or if the limits are one-sided.

Example 7.11 Let us see a limit that we already know, but obtained using l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Since $(\sin x)' = \cos x$ and $(x)' = 1$, and

$$\lim_{x\to 0} \frac{\cos x}{1} = 1,$$

we can readily conclude that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

■

Example 7.12 To calculate the limit

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{\sin x^2},$$

which is a $\frac{0}{0}$ indeterminacy, we first compute

$$\frac{d}{dx}(e^x - x - \cos x) = e^x - 1 + \sin x, \qquad \frac{d}{dx}\sin x^2 = 2x\cos x^2,$$

and try to obtain

$$\lim_{x\to 0} \frac{e^x - 1 + \sin x}{2x\cos x^2}.$$

This remains a $\frac{0}{0}$ indeterminacy, so again we compute

$$\frac{d}{dx}(e^x - 1 + \sin x) = e^x + \cos x, \qquad \frac{d}{dx}(2x\cos x^2) = 2\cos x^2 - 4x^2\sin x^2.$$

Now

$$\lim_{x\to 0} \frac{e^x + \cos x}{2\cos x^2 - 4x^2\sin x^2} = \frac{1+1}{2-0} = 1,$$

therefore

$$\lim_{x\to 0} \frac{e^x - 1 + \sin x}{2x\cos x^2} = 1$$

and finally

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{\sin x^2} = 1.$$

■

Exercise 7.5 Prove the equivalences, when $x \to 0$,

$$(1+x)^\alpha - 1 \sim \alpha x, \qquad e^x - 1 - x \sim \frac{x^2}{2}, \qquad x - \log(1+x) \sim \frac{x^2}{2}, \qquad x - \sin x \sim \frac{x^3}{6},$$

using l'Hôpital's rule. ■

> **R** From the definition of limit it is clear that
>
> $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell \quad \Rightarrow \quad \lim_{x\to\infty} \frac{f(n)}{g(n)} = \ell$$
>
> because $x_n = n$ is a particular sequence $x_n \to \infty$. Therefore l'Hôpital's rule can be applied to sequences as well.

There is an important caveat to be made about l'Hôpital's rule: in general, it is not true that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

as it is typically read in (incorrect) applications of this result. L'Hôpital's theorem states that the existence of the second limit implies that the first limit is the same, but it may well happen that the second limit does not exist while the first one does, as the following example illustrates.

■ **Example 7.13** Consider the limit

$$\ell = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}.$$

Since $\sin x \sim x$ when $x \to 0$, then

$$\ell = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin(1/x) = 0.$$

However, if we apply l'Hôpital's rule and try to calculate

$$\lim_{x\to 0} \frac{2x\sin(1/x) - \cos(1/x)}{\cos x}$$

this limit does not exist! ■

■ **Example 7.14** Before applying l'Hôpital's theorem one must check the hypothesis that the limits of the numerator and denominator are both 0 or $\pm\infty$. Otherwise the application of the theorem will lead to an incorrect result, as the following example illustrates:

$$\lim_{x\to 0} \frac{\cos x}{1 + \log(1+x)} = 1,$$

as it is easily verified. However, if one isists on applying l'Hôpital's theorem despite the fact that the limits of the numerator and denominator are neither 0 nor $\pm\infty$, this is what one would obtain:

$$\lim_{x\to 0} \frac{-\sin x}{\frac{1}{1+x}} = \lim_{x\to 0} \left[-(1+x)\sin x \right] = 0,$$

which is clearly wrong! ■

Problems

Problem 7.1 Let f and g be differentiable functions in $\mathbb{R}$. Write down the derivative of the following functions in their respective domains:

(i) $h(x) = \sqrt{f(x)^2 + g(x)^2}$;

(ii) $h(x) = \arctan\left(\dfrac{f(x)}{g(x)}\right)$;

(iii) $h(x) = f\big(g(x)\big)e^{f(x)}$;

(iv) $h(x) = \log\big(g(x)\sin f(x)\big)$;

(v) $h(x) = f(x)^{g(x)}$;

(vi) $h(x) = \dfrac{1}{\log\big(f(x) + g(x)^2\big)}$.

Problem 7.2

(a) Make up a continuous function in $\mathbb{R}$ which vanishes for $|x| \geqslant 2$ and equals 1 for $|x| \leqslant 1$.

(b) Do it again, but this time make sure that the function is differentiable in $\mathbb{R}$.

Problem 7.3 Check that the following functions satisfy the specified differential equations, where c, c_1, and c_2 are constants:

(i) $f(x) = \dfrac{c}{x}$ satisfies $xf' + f = 0$;

(ii) $f(x) = x\tan x$ satisfies $xf' - f - f^2 = x^2$;

(iii) $f(x) = c_1 \sin 3x + c_2 \cos 3x$ satisfies $f'' + 9f = 0$;

(iv) $f(x) = c_1 e^{3x} + c_2 e^{-3x}$ satisfies $f'' - 9f = 0$;

(v) $f(x) = c_1 e^{2x} + c_2 e^{5x}$ satisfies $f'' - 7f' + 10f = 0$;

(vi) $f(x) = \log(c_1 e^x + e^{-x}) + c_2$ satisfies $f'' + (f')^2 = 1$.

Problem 7.4 Prove the identities (valid only in the specified regions)

(i) $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}$, for $x > 0$;

(ii) $\arctan\dfrac{1+x}{1-x} - \arctan x = \dfrac{\pi}{4}$, for $x < 1$;

(iii) $2\arctan x + \arcsin\dfrac{2x}{1+x^2} = \pi$, for $x \geqslant 1$.

HINT: Differentiate the equation and check one point of the specified region.

Problem 7.5 At which points does the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent?

Problem 7.6 Given the function

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

calculate the angle between the tangents on the left and on the right of its graph at $x = 0$.

Problem 7.7 Find the sets where the function $f(x) = \sqrt{x+2}\arccos(x+2)$ is continuous and differentiable.

Problem 7.8 Calculate the smallest α for which $f(x) = |\alpha x^2 - x + 3|$ is differentiable in $\mathbb{R}$.

Problem 7.9 Given the function

$$f(x) = \begin{cases} a + bx^2, & |x| \leqslant c, \\ \dfrac{1}{|x|}, & |x| > c, \end{cases} \qquad c > 0,$$

find a and b so that it is continuous and differentiable in $\mathbb{R}$.

Problem 7.10 Given the function

$$f(x) = \begin{cases} \dfrac{3-x^2}{2}, & x < 1, \\ \dfrac{1}{x}, & x \geqslant 1, \end{cases}$$

(a) determine the sets where it is continuous and where it is differentiable;
(b) check that the mean value theorem can be applied to this function in $[0,2]$ by determining the point(s) $c \in (0,2)$ where the theorem holds.

Problem 7.11 Function $f(x) = 1 - x^{2/3}$ vanishes in $x = \pm 1$; however $f'(x) \neq 0$ in $(-1,1)$. Find which hypothesis of Rolle's theorem is not satisfied.

Problem 7.12 Prove, using Rolle's theorem, the following statements about a function f that is continuous in $[a,b]$ and differentiable in (a,b):
(i) If f vanishes $k(\geqslant 2)$ times in $[a,b]$ then f' vanishes at least $k-1$ times in $[a,b]$.
(ii) If f is n-times differentiable in (a,b) and vanishes in $n+1$ different points of $[a,b]$, then $f^{(n)}$ vanishes at least once in $[a,b]$.

Problem 7.13 Using the mean value theorem, find an approximation to $26^{2/3}$ and $\log(3/2)$.

Problem 7.14 Calculate the limits

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$;

(ii) $\displaystyle\lim_{x\to 0} \frac{\log|\sin 7x|}{\log|\sin x|}$;

(iii) $\displaystyle\lim_{x\to 1^+} \log x \log(x-1)$;

(iv) $\displaystyle\lim_{x\to\infty} x^{1/x}$;

(v) $\displaystyle\lim_{x\to 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$;

(vi) $\displaystyle\lim_{x\to\infty} x\left(\tan\frac{2}{x} - \tan\frac{1}{x}\right)$.

Problem 7.15 Suppose $h(x)$ is a twice-differentiable function and let

$$f(x) = \begin{cases} \dfrac{h(x)}{x^2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Calculate $h(0)$, $h'(0)$, and $h''(0)$ so that f is continuous.

Problem 7.16 Calculate the limits

(i) $\displaystyle\lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x - e\right]$;

(ii) $\displaystyle\lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^{x^2}}{e^x}$;

(iii) $\displaystyle\lim_{x\to\infty} \left(\frac{2^{1/x} + 18^{1/x}}{2}\right)^x$;

(iv) $\displaystyle\lim_{x\to\infty} \left(\frac{1}{p}\sum_{k=1}^{p} a_k^{1/x}\right)^x$, with $p \in \mathbb{N}$ and $a_k > 0$.

Problem 7.17 If f is a differentiable funtion such that

$$\lim_{x\to 0} \frac{f(2x^3)}{5x^3} = 1$$

and its derivative f' is continuous at $x = 0$,
(a) calculate $f(0)$;
(b) calculate $f'(0)$;

(c) calculate $\lim_{x\to 0} \frac{(f\circ f)(2x)}{f^{-1}(3x)}$.

Problem 7.18 The equation $e^{-f}f' = 2+\tan x$ together with the condition $f(0)=1$ define a one-to-one, differentiable function in the interval $[-\pi/4,\pi/4]$. If $g(x)=f^{-1}(x+1)$, calculate the limit

$$\lim_{x\to 0} \frac{e^x - e^{-\sin x}}{g(x)}.$$

Problem 7.19 Let $f(x) = |x^3(x-4)| - 1$.

(a) Find where f is continuous and where it is differentiable.

(b) Determine its extrema.

(c) Prove that $f(x)=0$ has a unique solution in $[0,1]$.

Problem 7.20 Solve these optimisation problems:

(a) A factory that produces tomato sauce wants to can it in cylindrical cans of a fixed volume V. Determine their radius r and height h so that their fabrication consumes the least possible material.

(b) A recipient with square bottom and no cap must be covered by a thin layer of lead. If the volume of the recipient must be 32 litres, which dimensions should it have so that it requires the least possible amount of lead?

(c) Find two numbers $x,y>0$ such that $x+y=20$ and x^2y^3 is maximum.

(d) Find the rectangle inscribed in the ellipse $(x/a)^2+(y/b)^2=1$ with its sides parallel to the axes of the ellipse, such that its area is maximum.

(e) With a tangent to the parabola $y=6-x^2$ and the positive axes one can make a triangle. Determine which of those triangles has the smallest area and compute it.

(f) We need to construct a box with no cap with the shape of a parallelepiped whose base is an equilateral triangle, and whose volume is 128 cm^3. If the material for the base costs 0.20 euros/cm^2 and that for the lateral surfaces costs 0.10 euros/cm^2, what are the dimensions of the cheapest such box?

(g) A right triangle ABC has vertex A at the origin, vertex B on the circumference $(x-1)^2+y^2 = 1$ —side AB is the hypothenuse of the triangle— and side AC on the horizontal axis. Calculate the location of C that maximises the area of the triangle.

(h) Let $\mathrm{P}=(x_0,y_0)$ be a point of the first quadrant $(x_0,y_0>0)$. A straight line through P cuts the axes at $\mathrm{A}=(x_0+\alpha,0)$ and $\mathrm{B}=(0,y_0+\beta)$. Calculate $\alpha>0$ and $\beta>0$ so as to minimise

(i) the length of segment AB;

(ii) the sum of the lengths of OA and OB;

(iii) the area of the triangle OAB.

HINT: Triangle similarity implies $\beta = x_0y_0/\alpha$.

Problem 7.21 Prove the following inequalities:

(a) $(1+x)^a \geqslant 1+ax$ for all $a\geqslant 1$, $x>-1$ (Bernoulli's inequality);

(b) $e^x \geqslant 1+x$ for all $x\in\mathbb{R}$;

(c) $\frac{x}{1+x} \leqslant \log(1+x) \leqslant x$ for all $x>-1$.

HINT: In all cases try to minimise the appropriate function.

Problem 7.22

(a) Prove that $\frac{\log x}{x} < \frac{1}{e}$ for all $x>0$, $x\neq e$.

(b) Prove that the previous inequality is equivalent to $e^x > x^e$ for all $x>0$, $x\neq e$.

Problem 7.23 Determine the number of solutions of the following equations in the specified domains:

(i) $x^7 + 4x = 3$ in $\mathbb{R}$;

(ii) $x^5 = 5x - 6$ in $\mathbb{R}$;

(iii) $x^4 - 4x^3 = 1$ in $\mathbb{R}$;

(iv) $\sin x = 2x - 1$ in $\mathbb{R}$;

(v) $x^x = 2$ in $[1, \infty)$;

(vi) $x^2 = \log \frac{1}{x}$ in $(1, \infty)$.