uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

All Units Final Exam.

Problem 1. **[1.5 points]** Prove that the following sequence is convergent and calculate its limit.

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \dfrac{1}{3-a_n} \end{cases}$$

Problem 2. **[2.5 points]** Study the convergence of the following series of real numbers:

a) **(1 pts)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{a^n\, n!}\ , \quad a < e$ **b)** **(1.5 pts)** $\displaystyle\sum_{n=1}^{\infty} \left(n \sin\frac{1}{n} \right)^{n^2}$

Problem 3. **[3.5 points]** Consider the function:

$$f(x) = \begin{cases} \dfrac{ax^2 + x + 1 - e^{bx}}{bx^2 + ax + a} & x \neq 0 \\ -1/2 & x = 0 \end{cases} \qquad ; \qquad a, b \in \mathbb{R}$$

a) **(0.75 pts)** Find the values of a and b that make f continuous in $\mathbb{R}$.

b) **(0.75 pts)** Calculate, if possible, the value of $f'(0)$ for the resulting function in part a).

c) **(1 pts)** Taking $a = 2$, $b = 0$ calculate, if possible, the Taylor polynomial of second degree for $f(x)$ centered at $x = -2$.

d) **(1 pts)** Give an upper bound for the error made when using the polynomial obtained in part c) to approximate $f(-3)$.

Problem 4. **[2.5 points]** Calculate the following integrals:

a) **(1 pts)** $\displaystyle\int_1^{64} \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ **b)** **(1.5 pts)** $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx$

SOLUTIONS

P1 (1 pt) Prove that the following sequence is convergent and calculate its limit.

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \dfrac{1}{3 - a_n} \end{cases}$$

SOLUTION. Supposing the sequence has a limit $a = \lim_{n\to\infty} a_n$, leads to

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{3 - \lim_{n\to\infty} a_n} \implies a = \frac{1}{3-a} \implies a^2 - 3a + 1 = 0 \implies a = \frac{3 \pm \sqrt{5}}{2}.$$

Now, looking at the first terms of the sequence, we have $\{1, 1/2, 2/5, 5/13, \dots\}$, so it would look as if the sequence is decreasing. But first, we will prove that $0 < a_n \leq 1$ for all $n \in \mathbb{N}$, using induction:

$a_n \leq 1$ It is true for $n = 1$. Supposing it is true for a_n, we have:

$$a_{n+1} = \frac{1}{3 - a_n} \leq \frac{1}{2} \leq 1,$$

which proves the result.

$a_n > 0$ Again it is true for $n = 1$ and, supposing it is true for a_n, we have

$$a_{n+1} = \frac{1}{3 - a_n} > \frac{1}{3} > 0,$$

which proves the result.

Now we can prove that a_n is decreasing:

$$a_{n+1} - a_n = \frac{1}{3 - a_n} - a_n = \frac{\left(\frac{3+\sqrt{5}}{2} - a_n\right)\left(a_n - \frac{3-\sqrt{5}}{2}\right)}{3 - a_n} \leq 0,$$

because as $0 < a_n \leq 1$, this implies $\left(\frac{3+\sqrt{5}}{2} - a_n\right) > 0$, $\left(a_n - \frac{3-\sqrt{5}}{2}\right) \leq 0$ and $3 - a_n > 0$.

Combining all these results, we have a decreasing sequence that is bounded below (by 0), which implies that the sequence is convergent. Because $a_1 = 1$ and the sequence is decreasing, the limit must be $a = \frac{3-\sqrt{5}}{2}$.

P2 (1.25 pts) Study the convergence of the following series of real numbers:

a) (0.50 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{a^n\, n!}$, $a < e$ **b) (0.75 pts)** $\displaystyle\sum_{n=1}^{\infty} \left(n \sin\frac{1}{n}\right)^{n^2}$

SOLUTION.

a) Using Stirling's approximation:

$$\lim_{n\to\infty} \frac{n^n}{a^n\, n!} = \lim_{n\to\infty} \frac{e^n}{a^n\, \sqrt{2\pi n}} = \infty,$$

as $e > a$. As the general term of the series does not converge to 0, we can say that the series is divergent.

b) We compute the limit of a_n:

$$\lim_{n\to\infty}\left(n\sin\frac{1}{n}\right)^{n^2}=\lim_{n\to\infty}\left(n\,\frac{1}{n}\right)^{n^2}=1^{\infty},$$

so the limit will be of the type e^c, where

$$c=\lim_{n\to\infty}n^2\left(n\sin\frac{1}{n}-1\right)=\lim_{n\to\infty}n^3\left(\frac{1}{n}-\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right)\right)-n^2=\lim_{n\to\infty}n^2-\frac{1}{6}+o(1)-n^2=-\frac{1}{6}.$$

This means that $a_n\to e^{-1/6}>0$, so the series is divergent.

P3 (2.5 pts) Consider the function:

$$f(x)=\begin{cases}\dfrac{ax^2+x+1-e^{bx}}{bx^2+ax+a} & x\neq 0\\[2ex] -1/2 & x=0\end{cases}\qquad;\qquad a,b\in\mathbb{R}$$

a) (0.75 pts) Find the values of a and b that make f continuous in $\mathbb{R}$.

b) (0.50 pts) Calculate, if possible, the value of $f'(0)$ for the resulting function in part *a*).

c) (0.75 pts) Taking $a=2$, $b=0$ calculate, if possible, the Taylor polynomial of second degree for $f(x)$ centered at $x=-2$.

d) (0.50 pts) Give an upper bound for the error made when using the polynomial obtained in part *c*) to approximate $f(-3)$.

SOLUTION.

a) In order for f to be continuous, we need $lim_{x\to 0}f(x)=-1/2$, so

$$\lim_{x\to 0}\frac{ax^2+x+1-e^{bx}}{bx^2+ax+a}=\lim_{x\to 0}\frac{0}{a}.$$

If $a\neq 0$, that limit would be zero, and f won't be continuous. However, if $a=0$, there is an indetermination, and using L'Hôpital's rule we have

$$\lim_{x\to 0}\frac{1-be^{bx}}{2bx}=\frac{1-b}{0},$$

in this case, in order for the limit to be finite we need $b=1$. Now using L'Hôpital's rule again

$$\lim_{x\to 0}\frac{-e^x}{2}=-\frac{1}{2},$$

and so f is continuous in $x=0$ if $a=0$ and $b=1$. The resulting function is

$$f(x)=\begin{cases}\dfrac{x+1-e^{x}}{x^2} & x\neq 0\\[2ex] -1/2 & x=0\end{cases}$$

Because it is a rational function whose denominator is zero only at $x=0$, we can say that f is continuous in $\mathbb{R}$.

b) Using the definition of the derivative, we have

$$f'(0) = \lim_{x\to 0} \frac{\frac{x+1-e^x}{x^2} + \frac{1}{2}}{x} = \lim_{x\to 0} \frac{2x+2-2e^x+x^2}{2x^3} = -\frac{1}{6}.$$

c) With $a = 2$ and $b = 0$, f becomes

$$f(x) = \begin{cases} \dfrac{2x^2+x}{2x+2} = x - \dfrac{1}{2} + \dfrac{1}{2(x+1)} & x \neq 0 \\ \\ -1/2 & x = 0 \end{cases}$$

For Taylor's polynomial, we need to compute

$$\begin{aligned} f(-2) &= -3, \\ f'(x) = 1 - \frac{1}{2(x+1)^2} \implies f'(-2) &= \frac{1}{2}, \\ f''(x) = \frac{1}{(x+1)^3} \implies f''(-2) &= -1. \end{aligned}$$

Therefore

$$P_{2,-2}(x) = -3 + \frac{x+2}{2} - \frac{(x+2)^2}{2}.$$

d) The error of the approximation is given by the formula

$$R_{2,-2}(-3) = \left| \frac{f'''(c)}{3!}(-3+2)^3 \right| = \frac{|f'''(c)|}{6},$$

with $c \in (-3, -2)$. Now,

$$f'''(x) = -\frac{3}{(x+1)^4},$$

which in the interval $(-3, -2)$ takes its maximum at $x = -2$:

$$\max_{x\in(-3,-2)} f'''(x) = f'''(-2) = -3,$$

which implies $R_{2,-2}(-3) \leq 1/2$.

P4 (1.25 pts) Calculate the following integrals:

a) **(0.50 pts)** $\displaystyle\int_1^{64} \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ **b)** **(0.75 pts)** $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

SOLUTION.

a) Using the change of variable $t^6 = x$ we have

$$\begin{aligned}\int_1^{64} \frac{dx}{\sqrt{x}+\sqrt[3]{x}} &= \int_1^{2} \frac{6t^5}{t^3+t^2}dt \\ &= 6\int_1^2 \left(t^2 - t + 1 - \frac{1}{t+1}\right) dt \\ &= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(t+1)\right)\Bigg|_1^2 = 11 + 6\log(2/3).\end{aligned}$$

b) We need to decompose the fraction into partial fractions:

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{Ax+B}{x^2+1}dx + \int \frac{C}{x-1}dx + \int \frac{D}{(x-1)^2}dx,$$

where $(Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1) = -2x+4$. Expanding the products and collecting the terms according to their degree, we have

$$(A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D) = -2x+4.$$

Because this equality has to hold for every x, the coefficients on both sides of the equation must be equal:

$$\begin{aligned} A + C &= 0 \\ -2A + B - C + D &= 0 \\ A - 2B + C = -2B - C + D &= 4 \end{aligned}$$

Solving this system yields $A = 2$, $B = 1$, $C = -2$ and $D = 1$. Therefore

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx - 2\int \frac{1}{x-1}dx + \int \frac{1}{(x-1)^2}dx \\ &= \log(x^2+1) + \arctan x - 2\log(x-1) - \frac{1}{x-1} + c.\end{aligned}$$