uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

All Units Final Exam.

Problem 1. [1.5 points] Show that the following sequence is convergent and calculate its limit.

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{2a_n + 3} \end{cases}$$

Problem 2. [2 points] Calculate the following limits, where $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

a) (1 pt) $\displaystyle \lim_{n\to\infty} \frac{\frac{2}{1} + \frac{3^2}{2} + \frac{4^3}{3^2} + \cdots + \frac{(n+1)^n}{n^{n-1}}}{n^2}$ **b) (1 pt)** $\displaystyle \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x \sin x} \right)$

Problem 3. [2.5 points] Let f be the following function:

$$f(x) = \begin{cases} \dfrac{a}{b\sin^2(x-1)+1} & x \in [0,1) \\ \arctan(x) - \log\left(\frac{x}{x-b}\right) & x \geq 1 \end{cases} \quad ; \quad a, b \in \mathbb{R}$$

a) (1 pt) Find (with proper justification) the relationship between a and b so that $f(x)$ is continous in $x = 1$.

b) (1 pt) Calculate, if possible, the values of a and b so that the resulting function in **a)** is differentiable in $x = 1$.

c) (0.50 pts) Graph schematically the function obtained in **b)**. (It is NOT necessary to find the exact values of any characteristic points of the function).

Problem 4. [2 points] Determine the area between the curves $y = x - 1$ and $x = 3 - y^2$.

Problem 5. [2 points] Calculate the following integrals:

a) (1 pt) $\displaystyle \int e^{2x} \sin x \, dx$ **b) (1 pt)** $\displaystyle \int \frac{3x+5}{x^3 - x^2 - x + 1} \, dx$

SOLUTIONS

Problem 1. [1.5 points] Show that the following sequence is convergent and calculate its limit.

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{2a_n + 3} \end{cases}$$

Solution. Supposing the sequence has a limit $a = \lim_{n\to\infty} a_n$, leads to

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2\lim_{n\to\infty} a_n + 3} \implies a = \sqrt{2a+3} \implies a^2 - 2a - 3 = 0 \implies a = 1 \pm 2 \implies a = -1 \text{ or } a = 3.$$

Now, looking at the first terms of the sequence, we have $1, \sqrt{5}, \sqrt{2\sqrt{5}+3} \approx \sqrt{7}, \ldots$, so it would look as if the sequence is increasing. But first, we will prove that $0 < a_n < 3$ for all $n \in \mathbb{N}$, using induction:

$a_n > 0$ It is true for $n = 1$. Supposing it is true for a_n, we have:

$$a_{n+1} = \sqrt{2a_n + 3} > \sqrt{3} > 0,$$

which proves the result.

$a_n < 3$ Again it is true for $n = 1$ and, supposing it is true for a_n, we have

$$a_{n+1} = \sqrt{2a_n + 3} < \sqrt{9} = 3,$$

which proves the result.

Now we can prove that a_n is increasing:

$$\frac{a_{n+1}}{a_n} > 1 \implies \frac{\sqrt{2a_n+3}}{a_n} > 1 \implies a_n^2 - 2a_n - 3 < 0 \implies (a_n+1)(a_n-3) < 0.$$

Now, because as $0 < a_n < 3$, the first term in the last inequality is always positive, while the second is always negative. This proves the result, and a_n is an increasing sequence.

Combining all these results, we have an increasing sequence that is bounded above by 3, which implies that the sequence is convergent. Because $a_1 = 1$ and the sequence is increasing, the limit must be $a = 3$. ■

Problem 2. [2 points] Calculate the following limits, where $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

a) (1 pt) $$\lim_{n\to\infty} \frac{\frac{2}{1} + \frac{3^2}{2} + \frac{4^3}{3^2} + \cdots + \frac{(n+1)^n}{n^{n-1}}}{n^2}$$ **b) (1 pt)** $$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x\sin x}\right)$$

Solution. **a)** Using Stolz's theorem:

$$\lim_{n\to\infty} \frac{\frac{2}{1} + \frac{3^2}{2} + \frac{4^3}{3^2} + \cdots + \frac{(n+1)^n}{n^{n-1}}}{n^2} = \lim_{n\to\infty} \frac{\frac{(n+1)^n}{n^{n-1}}}{n^2 - (n-1)^2} = \lim_{n\to\infty} \frac{n\left(\frac{n+1}{n}\right)^n}{2n-1} = \lim_{n\to\infty} \frac{n}{2n-1} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$
$$= \tfrac{e}{2}.$$

b) This is a $\infty - \infty$ indetermination:

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\sin x}\right)=\lim_{x\to 0}\frac{x\sin x - x^2}{x^2\sin x}=\lim_{x\to 0}\frac{x(x-x^3/6+o(x^3))-x^2}{x^2(x+o(x))}=\lim_{x\to 0}\frac{-x^3/6+o(x^3)}{x^3+o(x^3)}=-\frac{1}{6}.$$

■

Problem 3. [2.5 points] Let f be the following function:

$$f(x)=\begin{cases}\dfrac{a}{b\sin^2(x-1)+1} & x\in[0,1)\\[2ex] \arctan(x)-\log\left(\frac{x}{x-b}\right) & x\geq 1\end{cases}\quad ;\qquad a,b\in\mathbb{R}$$

- **a) (1 pt)** Find (with proper justification) the relationship between a and b so that $f(x)$ is continous in $x=1$.
- **b) (1 pt)** Calculate, if possible, the values of a and b so that the resulting function in **a)** is differentiable in $x=1$.
- **c) (0.50 pts)** Graph schematically the function obtained in **b)**. (It is NOT necessary to find the exact values of any characteristic points of the function).

__**Solution**__. **a)** In order for f to be continuous at $x=1$, we need $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$:

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{a}{b^2\sin(x-1)+1}=a,$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\arctan x-\log\left(\frac{x}{x-b}\right)=\frac{\pi}{4}-\log\left(\frac{1}{a-b}\right).$$

Therefore, the sought relationship is $a=\frac{\pi}{4}-\log\left(\frac{1}{a-b}\right)$.

b) In order for f to be differentiable at $x=1$, we need $\lim_{x\to 1^-} f'(x)=\lim_{x\to 1^+} f'(x)$:

$$\lim_{x\to 1^-} f'(x)=\lim_{x\to 1^-}\frac{-2ab\sin(x-1)\cos(x-1)}{(b\sin(x-1)+1)^2}=0,$$

$$\lim_{x\to 1^+} f'(x)=\lim_{x\to 1^+}\frac{1}{1+x^2}+\frac{b}{x(x-b)}=\frac{1}{2}+\frac{b}{1-b}.$$

From the two expressions above we obtain $b=-1$, and substituting in the expression we obtained in **a)**, we get $a=\frac{\pi}{4}+\log 2$.

c) f has a minimum at $x=1$: it is decreasing from $x=0$ to $x=1$ and increasing from $x=1$. And we also know that $f'(1)=0$. There is a horizontal asymptote when $x\to\infty$, at $\pi/2$. ■

Problem 4. [2 points] Determine the area between the curves $y=x-1$ and $x=3-y^2$.

Solution. The two curves cross at $(-2,-1)$ and $(2,1)$. It is easier to rotate the axes and integrate along the y axis. Thus, we want to find the area between the functions $f(y) = 3 - y^2$ and $g(y) = 1 + y$:

$$A = \int_{-2}^{1} [f(y) - g(y)]\, dy = \int_{-2}^{1} (2 - y - y^2)\, dy = \left(2y - y^2/2 - y^3/3\right)|_{-2}^{1} = \frac{9}{2}.$$

■

Problem 5. [2 points] Calculate the following integrals:

a) (1 pt) $\int e^{2x} \sin x\, dx$ **b) (1 pt)** $\int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx$

Solution. **a)** Integrating by parts:

$$\int e^{2x} \sin x\, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x\, dx = -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x\, dx.$$

Therefore, we have

$$\int e^{2x} \sin x\, dx = \frac{2^{2x}}{5} (-\cos x + 2 \sin x) + C.$$

b) This is a rational function. The roots of the denominator are 1 (double) and -1. Decomposing into partial frations, we obtain

$$\frac{3x+5}{x^3 - x^2 - x + 1} = -\frac{1}{2}\frac{1}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{2}\frac{1}{x+1}.$$

Therefore,

$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx = -\frac{1}{2}\log(x-1) - \frac{4}{x-1} + \frac{1}{2}\log(x+1) + C.$$

■