uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

All Units Final Exam.

Problem 1. [1.5 points] Show that the following sequence is convergent and find its limit.

$$a_1 = 1, \quad a_{n+1} = \log(1 + a_n)$$

HINT: Take into account the graphs of the functions $1 + x$ and e^x.

Problem 2. [1.5 points] Study the convergence of the following series:

a) (0.75 pts) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$ **b) (0.75 pts)** $\displaystyle\sum_{n=1}^{\infty} \sin(1+n)$

Problem 3. [1.5 points] Calculate the following limits:

a) (0.75 pts) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x \arctan x}{(\log(x+1))^2 \left(\sqrt[3]{x+1} - 1\right)}$ **b) (0.75 pts)** $\displaystyle\lim_{x\to 0} \frac{2e^{x^2} + \cos x - 3}{x^2}$

Problem 4. [1.5 points] Study the relative extrema of the following function in the interval $[1, \infty)$.

$$f(x) = \int_0^{x-1} \left(e^{-t^2} - e^{-2t} \right) dt$$

Problem 5. [2 points] Calculate the following integrals

a) (1 pts) $\displaystyle\int \frac{x^2}{x^2 - 4}\, dx$ **b) (1 pts)** $\displaystyle\int \frac{\log(\log x)}{x \log x}\, dx$

HINT: in b) use the change of variable $t = \log x$ repeatedly.

Problem 6. [2 points] Calculate the length of the curve given by

$$\begin{cases} x(t) = 2 + \dfrac{t^3}{3} \\ y(t) = \dfrac{t^2}{2} - 1 \end{cases}$$

with $0 \le t \le 1$.

SOLUTIONS

Problem 1. [1.5 points] Show that the following sequence is convergent and find its limit.

$$a_1 = 1, \quad a_{n+1} = \log(1 + a_n)$$

HINT: Take into account the graphs of the functions $1 + x$ and e^x.

Solution. We find the first terms: $\{1, \log 2, \log(1 + \log 2), \dots\} \implies$ looks like it is monotonously decreasing.

If the limits exists, it must be:

$$\ell = \log(1 + \ell) \quad ; \quad e^{\ell} = 1 + \ell \implies \ell = 0.$$

Now we show (using induction) that the sequence is bounded below by 0.

$$\begin{cases} a_1 = 1 > 0 \\ a_n > 0 \implies a_{n+1} = \log(1 + a_n) > 0 \iff a_n > 0 \end{cases}$$

We can now show that the sequence is monotonously decreasing:

$$\frac{a_{n+1}}{a_n} = \frac{\log(1 + a_n)}{a_n} < 1 \implies \log(1 + a_n) < a_n \implies 1 + a_n < e^{a_n},$$

which is true if $a_n \neq 0$, as in our case. Therefore, the sequence is decreasing and bounded by $0 < a_n \leq 1$, so it is convergent with limit 0. ■

Problem 2. [1.5 points] Study the convergence of the following series:

$$\textbf{a) (0.75 pts)} \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \quad \textbf{b) (0.75 pts)} \sum_{n=1}^{\infty} \sin(1 + n)$$

Solution. a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \sum_{n=1}^{\infty} \frac{2n+1-2n+1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{2}{4n^2-1}.$

We can use the limit comparison test:

$$\lim_{n \to \infty} \frac{\frac{1}{n^2}}{\frac{2}{4n^2-1}} = 2,$$

so the series is convergent as $\sum 1/n^2$ is convergent.

b) As $\lim_{n \to \infty} \sin(1 + n) \neq 0$, the series is divergent.

■

Problem 3. [1.5 points] Calculate the following limits:

$$\textbf{a) (0.75 pts)} \lim_{x \to 0} \frac{\sin^2 x \arctan x}{(\log(x+1))^2 \left(\sqrt[3]{x+1} - 1\right)} \quad \textbf{b) (0.75 pts)} \lim_{x \to 0} \frac{2e^{x^2} + \cos x - 3}{x^2}$$

Solution. a) Taking into account that $\sin^2 x = x^2 + o(x^2)$, $\arctan x = x + o(x)$, $\log(1+x) = x + o(x)$ and $\sqrt[3]{1+x} = x/3 + o(x)$, we have:

$$\lim_{x\to 0} \frac{\sin^2 x \arctan x}{(\log(x+1))^2 \left(\sqrt[3]{x+1} - 1\right)} = \lim_{x\to 0} \frac{x^3 + o(x^3)}{x^3/3 + o(x^3)} = 3.$$

b) We have $2e^{x^2} = 2 + 2x^2 + o(x^2)$, $\cos x = 1 - x^2/2 + o(x^2)$, and so

$$\lim_{x\to 0} \frac{2e^{x^2} + \cos x - 3}{x^2} = \lim_{x\to 0} \frac{2 + 2x^2 + 1 - x^2/2 - 3 + o(x^2)}{x^2} = \frac{3}{2}.$$

■

Problem 4. [1.5 points] Study the relative extrema of the following function in the interval $[1, \infty)$.

$$f(x) = \int_0^{x-1} \left(e^{-t^2} - e^{-2t}\right) dt$$

Solution. We obtain the derivative of f:

$$f'(x) = e^{-(x-1)^2} - e^{-2(x-1)}.$$

The relative extrema of f are the zeroes of f':

$$f'(x) = 0 \implies e^{-(x-1)^2} = e^{-2(x-1)} \implies (x-1)^2 = 2(x-1) \implies x = 1, 3.$$

Calculating the second derivative,

$$f''(x) = -2(x-1)e^{-(x-1)^2} + 2e^{-2(x-1)},$$

we have $f''(1) = 2 > 0$ and $f''(3) = -2e^{-4} < 0$, so $x = 1$ is a minimum and $x = 3$ is a maximum. ■

Problem 5. [2 points] Calculate the following integrals

a) (1 pts) $\displaystyle\int \frac{x^2}{x^2 - 4}\, dx$ **b) (1 pts)** $\displaystyle\int \frac{\log(\log x)}{x \log x}\, dx$

HINT: in b) use the change of variable $t = \log x$ repeatedly.

Solution. a) First, we note that

$$\frac{x^2}{x^2-4} = 1 + \frac{4}{x^2-4} = 1 + \frac{A}{x-2} + \frac{B}{x+2},$$

where $A(x+2) + B(x-2) = 4$. Solving this system leads to $A = 1, B = -1$, and so

$$\int \frac{x^2}{x^2-4}\, dx = \int \left(1 + \frac{1}{x-2} - \frac{1}{x+2}\right) dx = x + \log \frac{x-2}{x+2} + c.$$

b) We write $t = \log x$, and so $dt = dx/x$. Now,

$$\int \frac{\log(\log x)}{x \log x}\,dx = \int \frac{\log t}{t}\,dt.$$

We apply the same change of variable again: $u = \log t$, $du = dt/t$ and:

$$\int \frac{\log t}{t}\,dt = \int u\,du = \frac{u^2}{2} + c.$$

Undoing the change of variables, we obtain

$$\int \frac{\log(\log x)}{x \log x}\,dx = \frac{(\log(\log x))^2}{2} + c.$$

■

Problem 6. [2 points] Calculate the length of the curve given by

$$\begin{cases} x(t) = 2 + \dfrac{t^3}{3} \\ y(t) = \dfrac{t^2}{2} - 1 \end{cases}$$

with $0 \leq t \leq 1$.

Solution. The derivative of the curve is given by $x'(t) = t^2$, $y'(t) = t$. The length of the curve is

$$L = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^1 \sqrt{t^4 + t^2}\,dt = \int_0^1 t(1+t^2)^{1/2}\,dt = \left.\frac{(1+t^2)^{3/2}}{3}\right|_0^1 = \frac{2\sqrt{2}-1}{3}.$$

■