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Departamento de Matemáticas

CALCULUS I - OpenCourseWare

All Units Final Exam.

Problem 1. [**1 point**] Find the volume of the solid formed by revolving the region between the curves $\sqrt{x}$ and x around the y axis.

Problem 2. [**1.5 points**] Given the function $f(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$,

a) [**0.5 points**] Show that it is increasing in $(-\infty, \infty)$.

b) [**1 point**] Show that $y = e^{x^2}(1 + \sqrt{\pi} f(x))$ satisfies the differential equation $\dfrac{dy}{dx} - 2xy = 2$ and that $y(0) = 1$.

Problem 3. [**1 point**] Find $\displaystyle\lim_{x\to 0} \frac{xf(x)}{(f(x)-1)(x^2+f(x)^2)}$, considering that $\displaystyle\lim_{x\to 0} f(x) = 0$ and that f is twice differentiable.

HINT: Express the result in terms of $f'(0)$.

Problem 4. [**2 points**] Sum the following series: a) [**1 point**] $\displaystyle\sum_{n=1}^{\infty} \frac{n2^{n-3} + 5^n}{7^{n-1}}$

b) [**1 point**] $\displaystyle\sum_{n=2}^{\infty} (e^{1/n} - e^{1/(n-1)})$

Problem 5. [**2 points**] Calculate the following integrals:

a) [**1 point**] $\displaystyle\int_0^{\pi} x^2 \cos x\, dx$ b) [**1 point**] $\displaystyle\int \frac{4x}{(x^2+1)(x^2+2x+3)}\, dx$

Problem 6. [**2.5 points**] Given the recurrence:

$$a_{n+1} = 2 + \frac{3}{a_n}, \quad a_1 = 5/2,$$

a) [**0.5 points**] Show that $a_n > 2$ for all $n \in \mathbb{N}$.

b) [**0.5 points**] Supposing $a = \lim_{n\to\infty} a_n$ exists, find it.

c) [**1 point**] Show that the sequence is alternating, i.e $(a_{n+1} - a_n)(a_n - a_{n-1}) < 0$.

d) [**0.5 points**] Show that $|a_{n+1} - a| < \dfrac{1}{2}|a_n - a|$ and prove that the sequence is convergent.

SOLUTIONS

Problem 1. [**1 point**] Find the volume of the solid formed by revolving the region between the curves $\sqrt{x}$ and x around the y axis.

Solution. Let V_1 be the volume of the solid formed by revolving the area between the x axis and the curve $\sqrt{x}$ for $x \in [0,1]$, and V_2 the volume of the solid forme by revolving the are between the x axis and the curve x^2 for $x \in [0,1]$. Now, the volume that we want is

$$V = V_1 - V_2 = \int_0^1 2\pi x(\sqrt{x} - x)\, dx = 2\pi \left(\frac{2}{5} x^{5/2} - \frac{1}{3} x^3 \right) \Bigg|_0^1 = \frac{2}{15}\pi.$$

■

Problem 2. [**1.5 points**] Given the function $f(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$,
a) [**0.5 points**] Show that it is increasing in $(-\infty, \infty)$.
b) [**1 point**] Show that $y = e^{x^2}(1 + \sqrt{\pi} f(x))$ satisfies the differential equation $\dfrac{dy}{dx} - 2xy = 2$ and that $y(0) = 1$.

Solution. a) Applying the fundamental theorem of Calculus,

$$f'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2},$$

which is positive for all $x \in \mathbb{R}$. Applying the mean value theorem, this means that the funcion is monotonically increasing everywhere in $\mathbb{R}$.
b) Let us compute first

$$\frac{dy}{dx} = 2xe^{x^2}(1 + \sqrt{\pi} f(x)) + \sqrt{\pi} e^{x^2} f'(x) = 2xy + 2.$$

Now it is easy to check that y satisfies the differential equation. Finally, $y(0) = 1 + \sqrt{\pi} f(0) = 1$, as $f(0) = 0$. ■

Problem 3. [**1 point**] Find $\lim\limits_{x \to 0} \dfrac{xf(x)}{(f(x) - 1)(x^2 + f(x)^2)}$, considering that $\lim\limits_{x \to 0} f(x) = 0$ and that f is twice differentiable.

HINT: Express the result in terms of $f'(0)$.

Solution. If f is twice differentiable, then $\lim_{x \to 0} f(x) = 0$ implies $f(0) = 0$, and now we can express f as $f(x) = f'(0)x + o(x)$ (using its Taylor expansion at $x = 0$ up to first order). Now

$$\lim_{x \to 0} \frac{xf(x)}{(f(x) - 1)(x^2 + f(x)^2)} = -\lim_{x \to 0} \frac{f'(0)x^2 + o(x^2)}{x^2 + f'(0)^2 x^2 + o(x^2)} = -\frac{f'(0)}{1 + f'(0)^2}.$$

■

Problem 4. [2 points] Sum the following series: a) **[1 point]** $\sum_{n=1}^{\infty} \frac{n2^{n-3}+5^n}{7^{n-1}}$

b) **[1 point]** $\sum_{n=2}^{\infty}(e^{1/n}-e^{1/(n-1)})$

Solution. a) This is the combination of an arithmetic-geometric series and a geometric series:

$$\sum_{n=1}^{\infty} \frac{n2^{n-3}+5^n}{7^{n-1}} = \frac{7}{8}\sum_{n=1}^{\infty} n\left(\frac{2}{7}\right)^n + 7\sum_{n=1}^{\infty}\left(\frac{5}{7}\right)^n = \frac{7}{8}\frac{2/7}{(1-2/7)^2} + 7\frac{5/7}{1-5/7} = \frac{1799}{100}.$$

b) This is a telescoping series:

$$\sum_{n=2}^{\infty}(e^{1/n}-e^{1/(n-1)}) = \lim_{k\to\infty}\left(-e^{1/1}+e^{1/2}-e^{1/2}+\cdots+e^{1/(k-1)}-e^{1/(k-1)}+e^{1/k}\right) = -e+\lim_{k\to\infty} e^{1/k} = 1-e.$$

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Problem 5. [2 points] Calculate the following integrals:

a) **[1 point]** $\int_0^{\pi} x^2\cos x\,dx$ b) **[1 point]** $\int \frac{4x}{(x^2+1)(x^2+2x+3)}\,dx$

Solution. a) We integrate it by parts twice (first $u = x^2, dv = \cos x$, then $u = x, dv = \sin x$):

$$\int_0^{\pi} x^2\cos x\,dx = x^2\sin x\Big|_0^{\pi} - 2\int_0^{\pi} x\sin x\,dx = -2\left(-x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx\right) = -2\pi,$$

as $x^2\sin x\Big|_0^{\pi} = \int_0^{\pi}\cos x\,dx = 0$ and $-x\cos x\Big|_0^{\pi} = \pi$.

b) This is a rational function:

$$\int \frac{4x}{(x^2+1)(x^2+2x+3)}\,dx = \int \frac{Ax+B}{x^2+1}\,dx + \int \frac{Cx+D}{x^2+2x+3}\,dx,$$

as both factors in the denominator are irreducible. Now we have the equation

$$(Ax+B)(x^2+2x+3)+(Cx+D)(x^2+1) = 4x$$

and, by expanding the products and collecting the coefficients with the same degree, we obtain

$$(A+C)x^3+(2A+B+D)x^2+(3A+2B+C)x+(3B+D) = 4x.$$

This implies that

$$\begin{aligned} A+C&=0\\ 2A+B+D&=0\\ 3A+2B+C&=4\\ 3B+D&=0 \end{aligned}$$

Solving this system we obtain $A = B = 1$, $C = -1$, $D = -3$. That is

$$\int \frac{4x}{(x^2+1)(x^2+2x+3)}\,dx = \int \frac{x+1}{x^2+1}\,dx - \int \frac{x+3}{x^2+2x+3}\,dx.$$

Let us solve the first integral

$$\int \frac{x+1}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1) + \arctan x + c.$$

The second integral is

$$\begin{aligned} -\int \frac{x+3}{x^2+2x+3}\,dx &= -\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{2}{(x+1)^2+2}\,dx \\ &= -\frac{1}{2}\log(x^2+2x+3) - \sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c. \end{aligned}$$

Combining both results, we get

$$\int \frac{4x}{(x^2+1)(x^2+2x+3)}\,dx = \frac{1}{2}\log(x^2+1) + \arctan x - \frac{1}{2}\log(x^2+2x+3) - \sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + c.$$

■

Problem 6. [2.5 points] Given the recurrence:

$$a_{n+1} = 2 + \frac{3}{a_n}, \quad a_1 = 5/2,$$

a) [0.5 points] Show that $a_n > 2$ for all $n \in \mathbb{N}$.

b) [0.5 points] Supposing $a = \lim_{n\to\infty} a_n$ exists, find it.

c) [1 point] Show that the sequence is alternating, i.e $(a_{n+1} - a_n)(a_n - a_{n-1}) < 0$.

d) [0.5 points] Show that $|a_{n+1} - a| < \frac{1}{2}|a_n - a|$ and prove that the sequence is convergent.

Solution. a) We use induction. First, $a_1 = 5/2 > 2$. Now, assuming $a_n > 2$, we have

$$a_{n+1} = 2 + \frac{3}{a_n} > 2,$$

because $a_n > 0$ by the induction hypothesis.
b) Taking limits in both sides, we have

$$a = 2 + \frac{3}{a} \implies a^2 - 2a - 3 = 0 \implies a = 3 \text{ or } a = -1,$$

but we discard the negative value as $a_n > 0$ for all $n \in \mathbb{N}$. So if the limit exists, it has to be $a = 3$.
c) We write $a_{n-1} = 3/(a_n - 2)$ and substitute:

$$(a_{n+1} - a_n)(a_n - a_{n-1}) = \left(2 + \frac{3}{a_n} - a_n\right)\left(a_n - \frac{3}{a_n - 2}\right) = -\frac{(a_n^2 - 2a_n - 3)^2}{a_n(a_n - 2)}.$$

Now, the numerator is a square, so it is nonnegative. But we know the roots of this polynomial to be -1 and 3, and a_n never takes those values. The terms in the denominator are positive because $a_n > 2$ for all $n \in \mathbb{N}$. So this expression is negative.

d)

$$|a_{n+1} - a| = \left|2 + \frac{3}{a_n} - 3\right| = \left|\frac{3}{a_n} - 1\right| = \frac{1}{a_n}|a_n - 3| < \frac{1}{2}|a_n - a|,$$

as $1/a_n < 1/2$ for all $n \in \mathbb{N}$. This implies the convergence of the sequence, as the distance between a_n and a becomes increasingly smaller at each n, In more formal terms,

$$\lim_{n\to\infty} |a_n - a| < \lim_{n\to\infty} \frac{1}{2^{n-1}}|a_1 - a| = 0,$$

which proves the result. ■