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Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 1-4 Exam.

Problem 1. [2 points] Find (if they exist) the supremum, infimum, maximum and minimum of the set of $x \in \mathbb{R}$ that satify

$$\frac{|x^2-1|}{x} \geq 1+x.$$

Problem 2. [2 points] Let $a_1, a_2, \ldots, a_n \in (-1, 0]$. Using the induction principle, prove:

$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq 1+a_1+a_2+\cdots+a_n.$$

Problem 3. [2 points] Study the convergence of the following sequence, and find its limit if it exists:

$$\begin{cases} a_1 &= 1 \\ a_{n+1} &= \sqrt[3]{2a_n+(a_n)^2} \end{cases}$$

Problem 4. [2 points] Find, if they exist, the limit of the following sequences:

a) [1 point]

$$\lim_{n\to\infty} \frac{n^{n+\frac{3}{2}}\sqrt{2}}{e^n(n+1)!}$$

b) [1 point]

$$\lim_{n\to\infty} \left[\sqrt{n^2+2n}-n\right]^{n^2\left(1-\cos\frac{1}{\sqrt{n}}\right)}$$

Problem 5. [2 points] Study the convergence of the following series of real numbers:

$$\sum_{n=1}^{\infty} \frac{n^3+2\sqrt{n}}{\sqrt{n^7+3}}$$

SOLUTIONS

Problem 1. [2 points] Find (if they exist) the supremum, infimum, maximum and minimum of the set of $x \in \mathbb{R}$ that satify

$$\frac{|x^2-1|}{x} \geq 1+x.$$

Solution. There are two possible cases:

$|\mathbf{x}| \geq \mathbf{1}$. Then $|x^2-1| = x^2-1$, and

$$\frac{x^2-1}{x} \geq 1+x \iff \frac{x^2-1}{x} - 1 - x \geq 0 \iff \frac{1+x}{x} \leq 0.$$

If $x \geq 1$, both the numerator and the denominator are positive, and so the condition is not fulfilled. If, however, $x \leq -1$, the denominator is negative, and the numerator is never positive. But it becomes zero when $x = -1$, the only number that fulfills the condition.

$|\mathbf{x}| < \mathbf{1}$. Then $|x^2-1| = 1-x^2$ and

$$\frac{1-x^2}{x} \geq 1+x \iff \frac{1-x^2}{x} - 1 - x \geq 0 \iff \frac{1-x-2x^2}{x} \geq 0 \iff \frac{(x-1/2)(x+1)}{x} \leq 0.$$

If $-1 < x < 0$, both the numerator and the denominator are negative, and so no number in this interval fulfills the condition. If $x = 0$, this fraction is not defined, so the inequality does not make sense. If $0 < x \leq 1/2$, the denominator is positive and $(x-1/2)(x+1) \leq 0$, so the condition is fulfilled. If $x > 1/2$ both the numerator and the denominator are positive, and the condition is not fulfilled.

Joining both cases, we find that the set of numbers that fulfill the original condition is $\{-1\} \cup (0, 1/2]$. The infimum (and minimum) of this set is -1 and its supremum (and maximum) is $1/2$. ■

Problem 2. [2 points] Let $a_1, a_2, \ldots, a_n \in (-1, 0]$. Using the induction principle, prove:

$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

Solution. For $n = 1$ the inequality is clearly satisfied. Now, suppose it is true for a given n, and

$$\begin{aligned}(1+a_1)\ldots(1+a_n)(1+a_{n+1}) &\geq (1+a_1+\cdots+a_n)(1+a_{n+1}) \\ &= 1 + a_1 + \cdots + a_n + a_{n+1} + a_{n+1}a_1 + a_{n+1}a_2 + \cdots + a_{n+1}a_n \\ &\geq 1 + a_1 + a_2 + \cdots + a_n + a_{n+1},\end{aligned}$$

where we have used the induction hypothesis in the first line, expanded the product in the second and used the fact that all $a_n \in (-1, 0]$ in the third line: this last condition implies that $a_n a_m > 0$ for all $n, m \in \mathbb{N}$. ■

Problem 3. [2 points] Study the convergence of the following sequence, and find its limit if it exists:

$$\begin{cases} a_1 & = 1 \\ a_{n+1} & = \sqrt[3]{2a_n + (a_n)^2} \end{cases}$$

Solution. If this sequence had a limit l, it would have to satisfy the equation

$$l = \sqrt[3]{2l + l^2}.$$

The solutions of this equation are $l = 0, 1, 2$. As the first element in the sequence is $a_1 = 1$, we will show that the sequence is bounded by 2 above and by 1 below, using the induction principle:

$a_n \geq 1$**.** This is true for $n = 1$. Now, assuming it is valid for an arbitrary n, we have

$$a_{n+1} = \sqrt[3]{2a_n + (a_n)^2} \geq \sqrt[3]{3} \geq 1.$$

$a_n \leq 2$**.** This is true for $n = 1$. Now, assuming it is valid for an arbitrary n, we have

$$a_{n+1} = \sqrt[3]{2a_n + (a_n)^2} \leq \sqrt[3]{8} \leq 2.$$

Now let us check if the sequence is monotonic:

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt[3]{2a_n + (a_n)^2}}{a_n} = \sqrt[3]{2\frac{1}{a_n^2} + \frac{1}{a_n}} \geq \sqrt[3]{\frac{2}{4} + \frac{1}{2}} = 1,$$

where we have used the fact that $a_n^{-2} \geq 1/4$ and $a_n \geq 1/2$. This result implies that a_n is monotonically increasing. Because it is bounded above, it is convergent, and the only possibility for the limit is 2. ■

Problem 4. [2 points] Find, if they exist, the limit of the following sequences:

a) [1 point]

$$\lim_{n\to\infty} \frac{n^{n+\frac{3}{2}}\sqrt{2}}{e^n(n+1)!}$$

b) [1 point]

$$\lim_{n\to\infty} \left[\sqrt{n^2+2n} - n\right]^{n^2\left(1-\cos\frac{1}{\sqrt{n}}\right)}$$

Solution. **a)** Using Stirling's approximation, $n! \sim \sqrt{2\pi n}n^n e^{-n}$, we have

$$\lim_{n\to\infty} \frac{n^{n+\frac{3}{2}}\sqrt{2}}{e^n(n+1)!} = \lim_{n\to\infty} \frac{n^{n+\frac{3}{2}}\sqrt{2}}{e^n(n+1)\sqrt{2\pi n}n^n e^{-n}} = \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{\sqrt{\pi}}.$$

b) This is a limit of the type 1^∞. First we show that the limit of the base b_n is 1:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt{n^2+2n} - n = \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+2n}+n} = \lim_{n\to\infty} \frac{2}{\sqrt{1+2/n}+1} = 1.$$

Now we show that the exponent c_n grows to infinity with n:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} n^2 \left(1 - \cos\frac{1}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{n^2}{2n} = \frac{n}{2} = \infty,$$

where we have used the fact that $\cos\varepsilon_n - 1 \sim \varepsilon_n^2/2$.

Now, we know that limits of this type will be equal to e^c, where

$$c = \lim_{n\to\infty}(b_n - 1)c_n = \lim_{n\to\infty}\left(\sqrt{n^2+2n} - n - 1\right)\frac{n}{2} = \lim_{n\to\infty} -\frac{n}{2(\sqrt{n^2+2n}+n+1)} = -\frac{1}{4},$$

and therefore the limit is $e^{-1/4}$.

■

Problem 5. [2 points] Study the convergence of the following series of real numbers:

$$\sum_{n=1}^{\infty} \frac{n^3 + 2\sqrt{n}}{\sqrt{n^7+3}}$$

Solution. Using asymptotic approximations, we can show that

$$\frac{n^3 + 2\sqrt{n}}{\sqrt{n^7+3}} \sim \frac{n^3}{n^{7/2}} \sim \frac{1}{\sqrt{n}},$$

and so the series is divergent using the limit comparison criterion. ■