uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 1-4 Exam.

Problem 1. [2.5 points] Find the inverse of $f(x) = x^2 - x + 1$, assuming $x \geq 1/2$. What is the domain of $f^{-1}(x)$?

Problem 2. [2.5 points] Using the induction principle, prove:

$$(n+1)^2 + (n+2)^2 + (n+3)^3 + \cdots + (2n)^2 = \frac{n(2n+1)(7n+1)}{6}$$

Problem 3. [2.5 points] Consider the following sequence:

$$\begin{cases} a_0 & = 3 \\ a_{n+1} & = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \end{cases}$$

a) Suppose the sequence has a limit ℓ and compute its possible values.

b) Show that the sequence is bounded below by ℓ (HINT: show that $a_n - \ell \geq 0$).

c) Show that the sequence is monotonically decreasing.

d) Is the sequence convergent? Why? What is its limit?

Problem 4. [2.5 points] Sum the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n}$$

SOLUTIONS

Problem 1. [2.5 points] Find the inverse of $f(x) = x^2 - x + 1$, assuming $x \geq 1/2$. What is the domain of $f^{-1}(x)$?

Solution. The inverse $f^{-1}(x)$ is the function that satifies $f(f^{-1}(x)) = x$, that is

$$\left(f^{-1}(x)\right)^2 - f^{-1}(x) + 1 = x \implies f^{-1}(x) = \frac{1}{2} \pm \sqrt{x - \frac{3}{4}},$$

as the first equation is quadratic in $f^{-1}(x)$. Now, we are assuming that $x \geq 1/2$, and so the negative root of that equation is not valid in this domain. The inverse is therefore

$$f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}.$$

Now, in order for this function to make sense, the argument of the root has to be non-negative, i.e. $x \geq 3/4$. Therefore, the domain of the function is $[3/4, \infty)$. ■

Problem 2. [2.5 points] Using the induction principle, prove:

$$(n+1)^2 + (n+2)^2 + (n+3)^3 + \cdots + (2n)^2 = \frac{n(2n+1)(7n+1)}{6}$$

Solution. For $n = 1$ the equation reads

$$2^2 = \frac{1 \cdot 3 \cdot 8}{6} = 4,$$

which is true. Now, suppose the equation is true for a given n, and

$$\begin{aligned}(n+2)^2 + \cdots + (2n+2)^2 &= \left(\frac{n(2n+1)(7n+1)}{6} - (n+1)^2\right) + (2n+1)^2 + (2n+2)^2 \\ &= \frac{14n^3 + 51n^2 + 61n + 24}{6} \\ &= \frac{(n+1)(2(n+1)+1)(7(n+1)+1)}{6}\end{aligned}$$

which is what we wanted to prove. ■

Problem 3. [2.5 points] Consider the following sequence:

$$\begin{cases} a_0 &= 3 \\ a_{n+1} &= \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \end{cases}$$

a) Suppose the sequence has a limit ℓ and compute its possible values.

b) Show that the sequence is bounded below by ℓ (HINT: show that $a_n - \ell \geq 0$).

c) Show that the sequence is monotonically decreasing.

d) Is the sequence convergent? Why? What is its limit?

Solution. If this sequence had a limit ℓ, it would have to satisfy the equation

$$\ell = \frac{\ell}{2} + \frac{1}{\ell}.$$

The solutions of this equation are $\ell = \sqrt{2}, -\sqrt{2}$. Now, let us show that a_n is bounded below by $\sqrt{2}$:

$$a_{n+1} - \sqrt{2} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) - \sqrt{2} = \frac{a_n^2 + 2 - 2\sqrt{2}a_n}{2a_n} = \frac{(a_n - \sqrt{2})^2}{2a_n} \geq 0,$$

as the sequence is always positive (this is easy to show using induction). With this information, we can now show that the sequence is monotonically decreasing:

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) - a_n = \frac{2 - a_n^2}{2a_n} = \frac{(a_n + \sqrt{2})(a_n - \sqrt{2})}{2a_n} \leq 0.$$

Because this sequence is monotonically decreasing and it is bounded below, we know it will be convergent. The only possible value for the limit is $\ell = \sqrt{2}$. ■

Problem 4. [2.5 points] Sum the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n}$$

Solution. We can rewrite the sum as a telescoping series. Because

$$\frac{1}{n^2 + 4n} = \frac{1}{n(n+4)} = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+4}\right)$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n} &= \frac{1}{4}\left(\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+4}\right) \\ &= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \dots\right) \\ &= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{48}.\end{aligned}$$

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