uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 1-4 Exam.

Problem 1. [**3 points**] What is the domain of the function $f(x) = \arctan(\log(3x+5))$? Is f surjective? And injective? If so, find its inverse.

Problem 2. [**3 points**] Study the convergence of the following sequence, and find its limit if it exists:

$$a_{n+1} = \frac{1}{5}\left(a_n^2 + 6\right), \quad a_1 = \frac{3}{2}.$$

Problem 3. [**4 points**] Study the convergence of the following series of real numbers:

a) [**2 points**]

$$\sum_{n=1}^{\infty} \frac{n^3 + 2\sqrt{n}}{\sqrt{n^7 + 3}}$$

b) [**2 points**]

$$\sum_{n=1}^{\infty} \left(\frac{n^2 + n}{e^n}\right)^n$$

SOLUTIONS

Problem 1. [3 points] What is the domain of the function $f(x) = \arctan(\log(3x+5))$? Is f surjective? And injective? If so, find its inverse.

Solution. The domain of $\arctan(x)$ is the real line, but the logarithm needs its input to be positive. In other words, we need $3x + 5 > 0$. The domain of f is then $(-5/3, \infty)$.

The equation $y = \arctan(\log(3x + 5))$ does not have a solution for every y, as the range of $\arctan(x)$ is $(-\pi/2, \pi/2)$. So f is not surjective. Now, f is a composition of injective functions, so it is injective, and its inverse is

$$x = \arctan(\log(3f^{-1}(x) + 5) \iff \tan x = \log(3f^{-1}(x) + 5) \iff e^{\tan x} = 3f^{-1}(x) + 5,$$

so the inverse is $f^{-1}(x) = \dfrac{e^{\tan x} - 5}{3}$. ■

Problem 2. [3 points] Study the convergence of the following sequence, and find its limit if it exists:

$$a_{n+1} = \frac{1}{5}\left(a_n^2 + 6\right), \quad a_1 = \frac{3}{2}.$$

Solution. We assume that $a = \lim_{n\to\infty} a_n$ exists. Taking limits in the recurrence relation, we get $5a = a^2 + 6$, or equivalently $a^2 - 5a + 6 = 0$. Solving this equation, we obtain $a = 2$ or $a = 3$.

Now, we can prove that $a_n < 2$ for all natural numbers, using induction. For $n = 1$, we have that $a_1 = 3/2 < 2$. Assuming the inequality is true for a given n, we then have $a_{n+1} = (a_n^2 + 6)/5 < 10/5 = 2$, so this completes the proof.

We now need to check if the sequence is increasing or decreasing:

$$a_{n+1} - a_n = \frac{a_n^2 + 6}{5} - a_n = \frac{(a_n - 2)(a_n - 3)}{5} > 0$$

as both $a_n - 2 < 0$ and $a_n - 3 < 0$ for all $n \in \mathbb{N}$.

We have seen that a_n is a monotonically increasing sequence that is bounded above by 2, so it is convergent, and its limit is 2. ■

Problem 3. [4 points] Study the convergence of the following series of real numbers:

a) [2 points]

$$\sum_{n=1}^{\infty} \frac{n^3 + 2\sqrt{n}}{\sqrt{n^7 + 3}}$$

b) [2 points]

$$\sum_{n=1}^{\infty} \left(\frac{n^2 + n}{e^n}\right)^n$$

Solution. a) For large n, we have $n^3 + 2\sqrt{n} \sim n^3$ and $\sqrt{n^7+3} \sim n^{7/2}$, so $\dfrac{n^3 + 2\sqrt{n}}{\sqrt{n^7+3}} \sim \dfrac{1}{\sqrt{n}}$. Using the limit comparison test, the series is divergent.

b) Using the root test, we obtain

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n^2+n}{e^n} = 0$$

as $n^2 + n \prec e^n$, so the series is convergent. ∎