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Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 1-4 Exam.

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Problem 1. [2 points] Prove, using induction, that $n^3 + 5n$ is a multiple of 6 for every $n \in \mathbb{N}$.
(HINT: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$).

Problem 2. [2 points] What is the domain of the function $f(x) = \arcsin(\exp(x^2 - x - 20))$? Is f surjective? And injective? If so, find its inverse.

Problem 3. [3 points] Study the convergence of the following sequence, and find its limit if it exists:

$$a_{n+1} = \sqrt{6 + a_n}, \quad a_1 = 4.$$

Problem 4. [3 points] Study the convergence of the following series of real numbers:

a) [1.5 points]

$$\sum_{n=1}^{\infty} \frac{n^3 2^{n+3}}{7^{n-1}}$$

b) [1.5 points]

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

SOLUTIONS

Problem 1. [2 points] Prove, using induction, that $n^3 + 5n$ is a multiple of 6 for every $n \in \mathbb{N}$.
(HINT: $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$).

Solution. First, we check that the statement is true for $n = 1$. But in that case we have $n^3 + 5n = 6$, which is trivially a multiple of 6. Now, assuming the statement is true for some particular n, we have to check that it is true for the next natural number, $n + 1$. We have

$$(n + 1)^3 + 5(n + 1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n(n + 1) + 6.$$

Now, the first term $n^3 + 5n$ is a multiple of 6, because that's the induction hypothesis. The second term is the multiple of two consecutive natural numbers (and we know that number is a multiple of 2) times 3, so that is a multiple of 6. The last term is 6, so the right hand side is the sum of three terms that are all multiples of 6, which is a multiple of 6. This ends the proof. ■

Problem 2. [2 points] What is the domain of the function $f(x) = \arcsin(\exp(x^2 - x - 20))$? Is f surjective? And injective? If so, find its inverse.

Solution. The domain of $\arcsin(x)$ is $[-1, 1]$, so the domain of f will be all the numbers that make $-1 \leq \exp(x^2 - x - 20) \leq 1$. The exponential function is always positive, but it is only smaller than 1 if the argument is negative, that is, if $x^2 - x - 20 \leq 0$. Solving the inequality, we get $D(f) = [-4, 5]$.

The range of $\arcsin(x)$ is $[-\pi/2, \pi/2]$, so f is not surjective. It is also not injective: $f(-4) = f(5) = \arcsin(1) = \pi/2$. ■

Problem 3. [3 points] Study the convergence of the following sequence, and find its limit if it exists:

$$a_{n+1} = \sqrt{6 + a_n}, \quad a_1 = 4.$$

Solution. We assume that $a = \lim_{n\to\infty} a_n$ exists. Taking limits in the recurrence relation, we get $a = \sqrt{6 + a}$, or equivalently $a^2 - a - 6 = 0$. Solving the quadratic equation, we obtain two possible values for a: 3 and -2.

Now, we can prove that $a_n > 3$ for all natural numbers, using induction. For $n = 1$ we have $a_1 = 4 > 3$. Assuming the inequality is true for a given n, we then have $a_{n+1} = \sqrt{6 + a_n} > \sqrt{9} = 3$, which completes the proof.

We now need to check if the sequence is increasing or decreasing:

$$a_{n+1} - a_n = \sqrt{6 + a_n} - a_n = \frac{6 + a_n - a_n^2}{\sqrt{6 + a_n} + a_n} = \frac{(3 - a_n)(2 + a_n)}{\sqrt{6 + a_n} + a_n} < 0$$

as the last fraction is a product of one negative term $(3 - a_n)$ and two positive terms.

We have seen that a_n is a monotonically decreasing sequence that is bounded below by 3, so it is convergent, and its limit is 3. ■

Problem 4. [3 points] Study the convergence of the following series of real numbers:

a) [1.5 points]

$$\sum_{n=1}^{\infty} \frac{n^3 2^{n+3}}{7^{n-1}}$$

b) [1.5 points]

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$$

Solution. a) Using the quotient test, we get

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{n^3 2^{n+3} 7^{n-2}}{7^{n-1}(n-1)^3 2^{n+2}} = \lim_{n\to\infty} \frac{2}{7}\left(\frac{n}{n-1}\right)^3 = \frac{2}{7}$$

so the series is convergent.
b) We have that $e^{1/n} < 3$ for all $n \in \mathbb{N}$, so

$$\frac{e^{1/n}}{n^2} < 3\frac{1}{n^2}$$

But the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent, so using the comparison test we can say that $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ is convergent too. ∎