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Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 1-4 Exam.

Problem 1. [**2 points**] Prove, using induction, that $4^{2n+1} + 3^{n+2}$ is a multiple of 13 for every $n \in \mathbb{N}$.

Problem 2. [**2 points**] What is the domain of the function $f(x) = \log(\sinh(x^2 - x - 20))$? Is f surjective? And injective? If so, find its inverse.

Problem 3. [**3 points**] Study the convergence of the following sequence, and find its limit if it exists:

$$a_{n+1} = \frac{a_n^2}{1 + a_n}, \quad a_1 = 1.$$

Problem 4. [**3 points**] Study the convergence of the following series of real numbers:

a) [**1.5 points**]

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(2n)^n}$$

b) [**1.5 points**]

$$\sum_{n=1}^{\infty} \left(\frac{ne}{n+1}\right)^n$$

SOLUTIONS

Problem 1. [2 points] Prove, using induction, that $4^{2n+1} + 3^{n+2}$ is a multiple of 13 for every $n \in \mathbb{N}$.

Solution. First, we check that the statement is true for $n = 1$. But in that case we have $4^3 + 3^3 = 64 + 27 = 91$, which is 7 times 13. Now, assuming the statement is true for some particular n, we have to check that it is true for the next natural number, $n + 1$. We have

$$4^{2n+3} + 3^{n+3} = 16 * 4^{2n+1} + 3 * 3^{n+3} = 3(4^{2n+1} + 3^{n+2}) + 13 * 4^{2n+1}.$$

Now, the first term $3(4^{2n+1} + 3^{n+2})$ is a multiple of 13, because that's the induction hypothesis. The second term is 13 times a number, so the right hand side is the sum of two terms that are multiples of 13, which is a multiple of 13. This ends the proof. ■

Problem 2. [2 points] What is the domain of the function $f(x) = \log(\sinh(x^2 - x - 20))$? Is f surjective? And injective? If so, find its inverse.

Solution. The domain of $\log(x)$ is $(0, \infty)$, so the domain of f will be all the numbers that make $\sinh(x^2 - x - 20) > 0$. The hyperbolic sine is positive for positive arguments, so we need $x^2 - x - 20 > 0$. That is, $D(f) = (-\infty, -4) \cup (5, \infty)$.

The parabola $x^2 - x - 20$ maps the interval $(5, \infty)$ to $(0, \infty)$, and the hyperbolic sine maps $(0, \infty)$ to $(0, \infty)$. Finally, the logarithm maps $(0, \infty)$ to the real line, so f will be surjective.

However, f is not injective, because $x^2 - x - 20$ isn't: the parabola will map to different points x_1 and x_2 to the same value, and then $f(x_1) = f(x_2)$. For instance, $f(-5) = f(6) = \log(\sinh(10))$. ■

Problem 3. [3 points] Study the convergence of the following sequence, and find its limit if it exists:

$$a_{n+1} = \frac{a_n^2}{1 + a_n}, \quad a_1 = 1.$$

Solution. We assume that $a = \lim_{n\to\infty} a_n$ exists. Taking limits in the recurrence relation, we get $a = a^2/(1 + a)$, or equivalently $a + a^2 = a^2$. Solving this equation, we obtain $a = 0$.

Now, we can prove that $a_n > 0$ for all natural numbers, using induction. For $n = 1$ we have $a_1 = 1 > 0$. Assuming the inequality is true for a given n, we then have $a_{n+1} = a_n^2/(1 + a_n) > 0$, as all the terms in that fraction are positive. This completes the proof.

We now need to check if the sequence is increasing or decreasing:

$$a_{n+1} - a_n = \frac{a_n^2}{1 + a_n} - a_n = \frac{-a_n}{1 + a_n} < 0$$

as $a_n > 0$ for all $n \in \mathbb{N}$.

We have seen that a_n is a monotonically decreasing sequence that is bounded below by 0, so it is convergent, and its limit is 0. ■

Problem 4. [**3 points**] Study the convergence of the following series of real numbers:

a) [**1.5 points**]

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(2n)^n}$$

b) [**1.5 points**]

$$\sum_{n=1}^{\infty} \left(\frac{ne}{n+1}\right)^n$$

Solution. a) Using the quotient test, we get

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{(2n)!(n-1)!(2n-2)^{n-1}}{n!(2n)^n(2n-2)!} = \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)^{n-1}}{n(2n)^n} =$$

$$= \lim_{n\to\infty} 2\frac{2n-1}{2n-2}\left(\frac{2n-2}{2n}\right)^n = 2\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{2}{e}$$

so the series is convergent.

b) Using the root test, we obtain

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{ne}{n+1} = e$$

so the series is divergent. ∎