uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 5-8 Exam.

Problem 1. [2 points] Find the following limits:

a) [1 point] $\lim_{x\to 0} \dfrac{e^{2x} - \sin(2x) - 1}{x \tan x}$.   **b) [1 point]** $\lim_{x\to 0} \dfrac{x + \sin(\pi x)}{x - \sin(\pi x)}$.

Problem 2. [3 points] Given the function $f(x) = \begin{cases} \dfrac{x-1}{x^3} + \beta & \text{if } x > 1, \\ \arctan(\log x) & \text{if } x \le 1. \end{cases}$

a) [0.5 points] Find the domain of f.

b) [1.5 points] Find β so that f is differentiable in its domain.

c) [1 point] Taking $\beta = -1$, find the extrema of f.

Problem 3. [3 points] Given the function $f(x) = \log\left(\sqrt{1+x}\right)$

a) [1 point] Find its Taylor polynomial of degree 3 at $x = 0$.

b) [1 point] Use that polynomial to approximate the value of $\log\left(\sqrt{1.1}\right)$. Show that the error of the approximation is less than 10^{-4}.

c) [1 point] Find $f'(x)$ and compute its Taylor polynomial of degree 2 at $x = 0$.

Problem 4. [2 points] Given the series $S = \sum_{n=0}^{\infty} (-1)^n \dfrac{\pi^{2n+1}}{(2n+1)!}$

a) [1 point] Show that it is absolutely convergent.

b) [1 point] Compute its value (HINT: find a function whose Taylor series is similar to S).

SOLUTIONS

Problem 1. [2 points] Find the following limits:

a) [1 point] $\lim_{x\to 0} \dfrac{e^{2x} - \sin(2x) - 1}{x \tan x}$. **b) [1 point]** $\lim_{x\to 0} \dfrac{x + \sin(\pi x)}{x - \sin(\pi x)}$.

Solution. **a)** $\lim_{x\to 0} \dfrac{e^{2x} - \sin(2x) - 1}{x \tan x} = \lim_{x\to 0} \dfrac{1 + 2x + 2x^2 + o(x^2) - 2x + o(x^2) - 1}{x^2 + o(x^2)} = \lim_{x\to 0} \dfrac{2 + o(1)}{1 + o(1)} =$ 2,

where we have used the fact that $e^{2x} = 1 + 2x + 2x^2 + o(x^2)$, $\sin 2x = 2x + o(x^2)$, $\tan x = x + o(x^2)$ and $\lim_{x\to 0} o(1) = 0$.

b) $\lim_{x\to 0} \dfrac{x + \sin(\pi x)}{x - \sin(\pi x)} = \lim_{x\to 0} \dfrac{x + \pi x + o(x)}{x - \pi x + o(x)} = \lim_{x\to 0} \dfrac{1 + \pi + o(1)}{1 - \pi + o(1)} = \dfrac{1+\pi}{1-\pi}$,

where we have used the fact that $\sin x = x + o(x)$, and that $\lim_{x\to 0} o(1) = 0$.

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Problem 2. [3 points] Given the function $f(x) = \begin{cases} \dfrac{x-1}{x^3} + \beta & \text{if } x > 1, \\ \arctan(\log x) & \text{if } x \le 1. \end{cases}$

a) [0.5 points] Find the domain of f.

b) [1.5 points] Find β so that f is differentiable in its domain.

c) [1 point] Taking $\beta = -1$, find the extrema of f.

Solution. **a)** On the one hand, $(x-1)/x^3 + \beta$ is always defined for $x > 1$, as the sum of a rational function whose denominator is never 0 and a constant function. On the other hand, $\arctan(\log x)$ is defined only if x is positive. So the domain of f is $(0, \infty)$.

b) First we study the continuity of f. On the one hand, $(x-1)/x^3 + \beta$ is continuous for all $x > 1$ because the denominator x^3 is never 0 in this region. On the other hand, $\arctan(\log x)$ is continuous in $(0, 1)$ as it is the composition of two continuous functions. Now we need to check if f is continuous at $x = 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x-1}{x^3} + \beta = \beta,$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \arctan(\log x) = 0,$$

so, for f to be continuous at $x = 1$ (and therefore in all $\mathbb{R}$), β must be 0.

Now we study the differentiability of f: is f differentiable when $\beta = 0$? The derivative of f for $x \neq 1$ exists and is

$$f'(x) = \begin{cases} \dfrac{3-2x}{x^4} & \text{if } x > 1, \\ \dfrac{1}{x(1 + (\log x)^2)} & \text{if } x < 1. \end{cases}$$

Now we need to check if f is differentiable at $x = 1$:

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} \frac{3-2x}{x^4} = 1,$$
$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} \frac{1}{x(1+(\log x)^2)} = 1,$$

so for $\beta = 0$ we find that f is continuous and differentiable.

c) The only point that satisfies $f'(x) = 0$ is $x = 3/2$. We can check that $f'(x) > 0$ when $x < 3/2$ and $f'(x) < 0$ when $x > 3/2$, so it is a local maximum. But if $\beta = -1$, f is not continuous at $x = 1$, so we need to check the value of f there. We find that $f(1) = \arctan(\log 1) = 0$. Now, because $f'(x) > 0$ for $x \in (0, 1)$ and $\lim_{x\to 1^+} f(x) = \beta = -1$, we find that $x = 1$ is a global maximum.

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Problem 3. [3 points] Given the function $f(x) = \log\left(\sqrt{1+x}\right)$

a) [1 point] Find its Taylor polynomial of degree 3 at $x = 0$.

b) [1 point] Use that polynomial to approximate the value of $\log\left(\sqrt{1.1}\right)$. Show that the error of the approximation is less than 10^{-4}.

c) [1 point] Find $f'(x)$ and compute its Taylor polynomial of degree 2 at $x = 0$.

<u>**Solution**</u>. **a)** We have

$$f(x) = \frac{1}{2}\log(1+x) = \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right),$$

and so $P_{3,0}(x) = \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$.

b) We need to find $P_{3,0}(0.1) = 1/20 - 1/400 + 1/6000$. The remainder of this Taylor polynomial is

$$R_{3,0}(x) = \left|\frac{f^{iv}(c)}{4!}x^4\right|,$$

where $c \in (0, x)$. Now, $f^{iv}(x) = -3(1+x)^{-4}$, and so we have

$$R_{3,0}(0.1) = \frac{(10^{-1})^4}{8(1+c)^4} \le \frac{1}{8}10^{-4} < 10^{-4}.$$

c) The derivative is $f'(x) = 1/(2+2x)$, and its Taylor expansion of second degree is simply the derivative of $P_{3,0}(x)$: $\dfrac{1-x+x^2}{2}$.

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Problem 4. [2 points] Given the series $S = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$

a) [1 point] Show that it is absolutely convergent.

b) **[1 point]** Compute its value (HINT: find a function whose Taylor series is similar to S).

<u>Solution</u>. **a)** We need to show that

$$\sum_{n=0}^{\infty}\left|(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}\right| = \sum_{n=0}^{\infty}\frac{\pi^{2n+1}}{(2n+1)!}$$

is convergent. Using the quotient test

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\pi^{2n+3}(2n+1)!}{(2n+3)!\pi^{2n+1}} = \lim_{n\to\infty}\frac{\pi^2}{(2n+2)(2n+3)} = 0,$$

and so the series is absolutely convergent.

b) The Taylor series for $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

so if we take $x = \pi$ we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!} = \sin\pi = 0.$$

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