uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 5-8 Exam.

Problem 1. [4 points] Consider the function

$$f(x) = \begin{cases} \log\left(\frac{1}{|x|}\right), & \text{if } |x| > 1, \\ a\cos(b\pi x), & \text{if } |x| \le 1 \end{cases}$$

a) Find (if possible) the values of a and b that make f continuous and differentiable everywhere. [**1 point**]

b) Using $a = 1$,$b = 1$, find the relative maxima and minima of f. [**1 point**]

c) Using $a = 1$, $b = 1$, find the sets where f is convex. [**1 point**]

d) Draw a rough sketch of the function using the information obtained above (and either the values of a and b obtained in part a) or those in part b). [**1 point**]

Problem 2. [4 points]

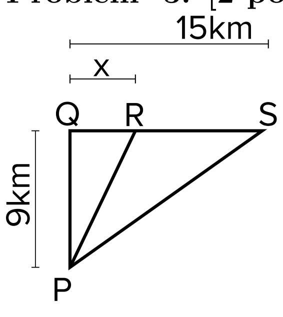
a) Calculate the integral $\displaystyle\int_0^1 \cos^2 x\, dx$. [**1 point**]

b) Find the Taylor polynomial of degree 5 for $\cos^2 x$, centered at $x = 0$. [**1 point**]

c) Use the polynomial in part b) to approximate the integral in part a). [**1 point**]

d) What is the error made by this approximation? (HINT: find the error $R(x)$ of your Taylor approximation using the formula for the remainder. Then integrate $\displaystyle\int_0^1 R(x)\, dx$). [**1 point**]

Problem 3. [2 points]



A woman is on an island at point P and wants to reach point S in the shore. In order to reach the shore she will use a boat, rowing at a speed of 3km/h. She will land at point R, and from R to S she will walk at a speed of 5 km/h. Where should she land her boat so that she takes the least time to reach S?

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Solution. a) Both the logarithm and the cosine are continuous in their domains, and f is an even function, so we just need to check that the function is continuous at $x = 1$:

$$\lim_{x\to 1^-} a\cos(b\pi x) = 0 = \lim_{x\to 1^+} \log(1/x),$$

This implies that $a\cos(b\pi) = 0$, which in turn implies that either $a = 0$ or $b = (2k+1)/2$, with $k \in \mathbb{Z}$. in order for f to be continuous.

Now the derivative of f is

$$f'(x) = \begin{cases} -1/x, & \text{if } |x| > 1, \\ -ab\pi\sin(b\pi x), & \text{if } |x| < 1 \end{cases}$$

Again, both pieces of f' exist in their respective domains, so we just need to check that $f'(1)$ exists:

$$\lim_{x\to 1^-} -ab\pi\sin(b\pi x) = -1 = \lim_{x\to 1^+} -\frac{1}{x},$$

which implies that $ab\pi\sin(b\pi) = 1$. But if f is to be differentiable, it must be continuous. So either $a = 0$ or $b = (2k+1)/2$, from above. Now, if $a = 0$ we have $0 = 1$, so it must be $a \neq 0$ and $b = (2k+1)/2$. But then we have $\sin(b\pi) = \pm 1$ and $a = \pm(b\pi)^{-1}$, with a being positive if k is even, and negative if k is odd. (In order to get full marks in this exercise, it was enough to arrive at $b = 1/2$ and $a = 2/\pi$).

b) The derivative of $\log(1/|x|)$ is $-1/x$, so $\log(1/x)$ is increasing when $x < 0$ and decreasing when $x > 0$, and there are no maxima nor minima in $|x| > 1$. Now, $\cos(\pi x)$ has maxima at $x = 2k$ and minima at $x = 2k+1$, where $k \in \mathbb{Z}$. Since the part of f that is a cosine is just between $-1 \leq x \leq 1$, f has a relative maximum at $x = 0$ and two relative minima at $x = -1$ and $x = 1$.

c) Using $a = 1$, $b = 1$:

$$f''(x) = \begin{cases} 1/x^2, & \text{if } |x| > 1, \\ -\pi^2\cos(\pi x), & \text{if } |x| < 1 \end{cases}$$

In $(-\infty, -1) \cup (1, \infty)$ the second derivative is positive, so f is convex in this set. In $(-1, 1)$, the second derivative is positive in $(-1, -1/2) \cup (1/2, 1)$. In summary, we have that f is convex in $(-\infty, -1/2) \cup (-1/2, \infty)$. ■

Problem 2. [4 points]

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b) Find the Taylor polynomial of degree 5 for $\cos^2 x$, centered at $x = 0$. **[1 point]**

c) Use the polynomial in part b) to approximate the integral in part a). **[1 point]**

d) What is the error made by this approximation? (HINT: find the error $R(x)$ of your Taylor approximation using the formula for the remainder. Then integrate $\int_0^1 R(x)\, dx$). **[1 point]**

Solution. a) Since $\cos^2 x = (1 + \cos 2x)/2$, we have

$$\int_0^1 \cos^2 x\, dx = \frac{1}{2}\int_0^1 (1 + \cos 2x)dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)|_0^1 = \frac{1}{2} + \frac{\sin 2}{4}.$$

b) Since $\cos 2x = 1 - 2x^2 + 2x^4/3 + o(x^5)$, we have

$$P_{5,0}(x) = \frac{1}{2}(1 + 1 - 2x^2 + 2x^4/3) = 1 - x^2 + \frac{1}{3}x^4.$$

c)

$$\int_0^1 \cos^2 x\, dx \approx \int_0^1 \left(1 - x^2 + \frac{1}{3}x^4\right) dx = \left(x - \frac{x^3}{3} + \frac{x^5}{15}\right)|_0^1 = 1 - \frac{1}{3} + \frac{1}{15} = \frac{11}{15}.$$

d) The remainder is given by

$$R_{5,0}(x) = \left|\frac{f^{(vi)}(c)}{6!}x^6\right|,$$

where $c \in (0,1)$. The sixth derivative is $f^{(vi)}(x) = -32\cos 2x$ and its maximum value in $(0,1)$ is 1. So

$$R_{5,0}(x) \le \frac{32}{720}x^6.$$

Therefore, the error of the integral will be

$$\int_0^1 R_{5,0}(x) \le \int_0^1 \frac{32}{720}x^6\, dx = \frac{32}{5040} \approx 6.3 \cdot 10^{-3}.$$

The actual error is $6 \cdot 10^{-3}$. ■

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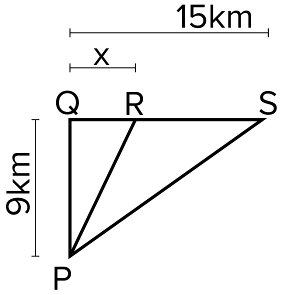

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Solution. The time she takes to row is $T_r = D_r/3$, and the time she takes to walk is $T_w = D_w/5$, where D_r and D_w are the distances she needs to row and walk, respectively.

Now, if she chooses point R at some distance x from point Q, the distance she will row is $\sqrt{9^2 + x^2}$, while the distance she will wlak is $15 - x$. Therefore, the function we need to minimize is

$$T(x) = \frac{\sqrt{81 + x^2}}{3} + \frac{15 - x}{5}.$$

This is a differentiable function everywhere, so its relative minima will be given by the points where $T'(x) = 0$:

$$T'(x) = \frac{x}{3\sqrt{81 + x^2}} - \frac{1}{5} = \frac{5x - 3\sqrt{81 + x^2}}{15\sqrt{81 + x^2}} = 0.$$

Now, we multiply and divide by $5x + 3\sqrt{81 + x^2}$ and we obtain

$$T'(x) = 0 \implies 25x^2 - 9(81 + x^2) = 0 \implies x^2 = \frac{3^6}{16} \implies x = \frac{27}{4}.$$

It's easy to check that if $x < 27/4$, $f'(x) < 0$ and if $x > 27$, $f'(x) = 0$, so this is indeed a minimum. ■