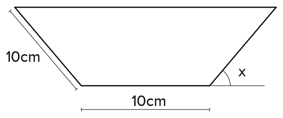

Problem 1. [2 points]



A pipe with a trapezoid section has the dimensions shown in the figure. Determine the value of the angle x (between 0 and π) so that the volume that the pipe carries is maximal. (HINT: $\cos(\pi \pm x) = -\cos x$)

Problem 2. [3 points] Consider the function $f(x) = \arctan\left(\dfrac{1}{1-x}\right)$

a) Find the sets where it is continuous and differentiable. Find its relative maxima and minima, and the sets where it is convex. **[1.5 points]**

b) Draw a rough sketch of the function using the information obtained above. **[0.5 points]**

c) Find the third degree Taylor polynomial of the function centered at $x = 0$. **[1 point]**

Problem 3. [3 points]

a) Approximate the value of $\dfrac{1}{\sqrt[3]{9}}$ using the second degree Taylor expansion of an appropriate function. **[1.5 points]**

b) What is the (approximate) maximum value of the error made in this approximation? **[1 point]**

c) What should be the degree of the approximation in order for the error to be smaller than 10^{-4}? **[0.5 points]**

Problem 4. [2 points] Calculate the following limits:

a) [1 point]

$$\lim_{x\to 0} \frac{\sin^2 x - (e^x - 1)^2}{\log(1 + x^3)}$$

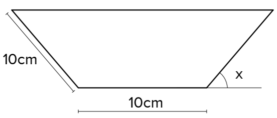
b) [1 point]

$$\lim_{x\to\infty} x^2 \left(\sqrt{1 - 1/x} - x\log(1 + 1/x)\right)$$

(HINT: $\sqrt{1-z} = 1 - z/2 - z^2/8 + o(z^2)(z \to 0)$)

SOLUTIONS

Problem 1. [2 points]



A pipe with a trapezoid section has the dimensions shown in the figure. Determine the value of the angle x (between 0 and π) so that the volume that the pipe carries is maximal. (HINT: $\cos(\pi \pm x) = -\cos x$)

Solution. The maximal volume is reached when the area of the section is maximal as well. Now, the area of the section is the sum of the area of the rectangle in the middle, that is $10h$ (where h is the height of the section), and the areas of the two triangles in the sides, which are equal to $bh/2$ (where b is the base of the triangle). So the area that we want to maximize is $A = 10h + bh$. Now, $h = 10 \sin x$ and $b = 10 \cos x$, so

$$A(x) = 100 \sin x + 100 \sin x \cos x = 100 \left(\sin x + \frac{1}{2} \sin 2x \right).$$

This is a smooth function, so its relative maxima will be given by the zeroes of its derivative:

$$A'(x) = 100(\cos x + \cos 2x) = 0 \implies \cos x = -\cos 2x.$$

If $\cos z = -\cos y$ that means that $z = \pi - y$ or $z = \pi + y$. In the first case, substituting we obtain $2x = \pi - x$, which leads to $x = \pi/3$, and in the second case we obtain $2x = \pi + x$, which leads to $x = \pi$. It's obvious that if $x = \pi$ the area of the section will be zero, so the only possibility for the maximum is $x = \pi/3$ (we can also check that the second derivative is negative at this value). ■

Problem 2. [3 points] Consider the function

$$f(x) = \arctan\left(\frac{1}{1-x}\right)$$

a) Find the sets where it is continuous and differentiable. Find its relative maxima and minima, and the sets where it is convex. **[1.5 points]**

b) Draw a rough sketch of the function using the information obtained above. **[0.5 points]**

c) Find the third degree Taylor polynomial of the function centered at $x = 0$. **[1 point]**

Solution. a) The arc tangent is continuous and differentiable everywhere. The function $1/(1-x)$ is continuous and differentiable everywhere, except at $x = 1$, where it is not defined. As a result, f will be continuous and differentiable everywhere except at $x = 1$.

The derivative of f is

$$f'(x) = \frac{1}{(x-1)^2 + 1},$$

which is always positive. As a result, f has no maxima nor minima, and it is monontonically increasing.

The second derivative is

$$f''(x) = -\frac{2(x-1)}{((x-1)^2+1)^2}$$

which is positive when $x < -1$ and negative if $x > 1$. That is, f is convex if $x < 1$ and concave if $x > 1$.

c) The third derivative of f is

$$f'''(x) = \frac{2(3x^2-6x+2)}{((x-1)^2+1)^3}$$

which, evaluated at $x = 0$ gives $f'''(0) = 1/2$. From part a) we know that $f(0) = \pi/4$, $f'(0) = f''(0) = 1/2$. The Taylor polynomial of third degree centered at $x = 0$ is

$$P_{3,0}(x) = \frac{\pi}{4} + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12}.$$

■

Problem 3. [3 points]

a) Approximate the value of $\frac{1}{\sqrt[3]{9}}$ using the second degree Taylor expansion of an appropriate function. **[1.5 points]**

b) What is the (approximate) maximum value of the error made in this approximation? **[1 point]**

c) What should be the degree of the approximation in order for the error to be smaller than 10^{-4}? **[0.5 points]**

Solution. a) The second degree Taylor polynomial of the function $f(x) = (8+x)^{-1/3}$ is

$$P_{2,0}(x) = \frac{1}{2} - \frac{x}{48} + \frac{x^2}{576}$$

which, evaluated at $x = 1$, yields

$$P_{2,0}(1) = \frac{1}{2} - \frac{1}{48} + \frac{1}{576} = \frac{277}{576} \approx 0.48090277777\ldots$$

b) The remainder of the previous polynomial is given by

$$R_{2,0}(x) = \left|\frac{f'''(c)}{6}x^3\right|$$

where $c \in (0, x)$. The third derivative of f is

$$f'''(x) = -\frac{28}{27(8+x)^{10/3}}$$

whose absolute value is a decreasing function in $(0, 1)$ and therefore has a maximum value at $x = 0$. As a result,

$$R_{2,0}(1) = \left|\frac{f'''(c)}{6}\right| < \left|\frac{f'''(0)}{6}\right| = \frac{28}{6 \cdot 27 \cdot 2^{10}} \approx 1.6 \cdot 10^{-4}.$$

The actual error made is actually $1.5 \cdot 10^{-4}$.
c) The next remainder is

$$R_{3,0}(1) < \left| \frac{f^{(iv)}(0)}{24} \right| = \frac{280}{24 \cdot 81 \cdot 2^{13}} = \frac{35}{1990656} \approx 1.7 \cdot 10^{-5},$$

so that will do. ■

Problem 4. **[2 points]** Calculate the following limits:

a) [1 point]

$$\lim_{x \to 0} \frac{\sin^2 x - (e^x - 1)^2}{\log(1 + x^3)}$$

b) [1 point]

$$\lim_{x \to \infty} x^2 \left(\sqrt{1 - 1/x} - x \log(1 + 1/x) \right)$$

(HINT: $\sqrt{1-z} = 1 - z/2 - z^2/8 + o(z^2)(z \to 0)$)

Solution. a) Using the Taylor expansions of the elementary functions in this expression up to second degree, we obtain

$$\lim_{x \to 0} \frac{(x + o(x^2))^2 - (x + x^2/2 + o(x^2))^2}{x^3 + o(x^3)} = \lim_{x \to 0} \frac{-x^3 + o(x^3)}{x^3 + o(x^3)} = -1$$

b) First we change the variable $t = 1/x$, then

$$\lim_{t \to 0} \frac{\sqrt{1-t} - \log(1+t)/t}{t^2} = \lim_{t \to 0} \frac{(1 - t/2 - t^2/8 + o(t^2)) - (1 - t/2 + t^2/3 + o(t^2))}{t^2} = -\frac{11}{24}$$

■