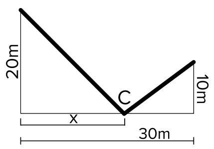
CALCULUS I - OpenCourseWare

Units 5-8 Exam.

Problem 1. [**2 points**]



A cable is bound to two posts 30 meters apart, and it is held fixed to the floor at a point C between them (see figure). The height of the leftmost post is 20 meters, while the height of the rightmost one is 10 meters. What is the distance x from the leftmost post at which we have to bind the cable so that its length is minimal?

Problem 2. [**3 points**] Consider the function $f(x) = \begin{cases} |x|, & \text{if } |x| < 1, \\ \frac{1}{1+(x+1)^2}, & \text{if } |x| \geq 1 \end{cases}$

a) Find the sets where it is continuous and differentiable. Find its relative maxima and minima, and the sets where it is convex. [**1.5 points**]

b) Draw a rough sketch of the function using the information obtained above. [**0.5 points**]

c) Find the second degree Taylor polynomial of the function centered at $x = 2$. [**1 point**]

Problem 3. [**3 points**]

a) Approximate the value of $\log 0.9$ using the third degree Taylor expansion of an appropriate function. [**1.5 points**]

b) What is the (approximate) maximum value of the error made in this approximation? [**1 point**]

c) What should be the degree of the approximation in order for the error to be smaller than 10^{-5}? [**0.5 points**]

Problem 4. [**2 points**] Calculate the following limits:

a) [**1 point**]

$$\lim_{x\to 0} \frac{\sin x \cos x - \tan x}{x^2 \log(1+x)}$$

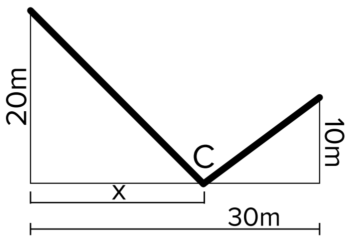
(HINT: $\tan x = x + x^3/3 + o(x^3)$).

b) [**1 point**]

$$\lim_{x\to\infty} \frac{\cos(\frac{1}{x}) - 1 - \frac{1}{2x^2}}{\sin(\frac{1}{6x^2})}$$

SOLUTIONS

Problem 1. [2 points]



A cable is bound to two posts 30 meters apart, and it is held fixed to the floor at a point C between them (see figure). The height of the leftmost post is 20 meters, while the height of the rightmost one is 10 meters. What is the distance x from the leftmost post at which we have to bind the cable so that its length is minimal?

Solution. The length of the left-hand cable is $\sqrt{20^2+x^2}$, while the right-hand cable is $\sqrt{(30-x)^2+10^2}$ meters long. So the function we need to minimize is

$$R(x) = \sqrt{20^2+x^2} + \sqrt{(30-x)^2+10^2}.$$

Its derivative is

$$R'(x) = \frac{x}{\sqrt{20^2+x^2}} - \frac{30-x}{\sqrt{(30-x)^2+10^2}} = \frac{x\sqrt{(30-x)^2+10^2} - (30-x)\sqrt{20^2+x^2}}{A(x)},$$

where $A(x)$ is a function of x that is always positive, so we can neglect it when looking for zeros of $R'(x)$. Now, we need to find the solutions of this equation

$$x\sqrt{(30-x)^2+10^2} - (30-x)\sqrt{20^2+x^2} = 0.$$

Multiplying and dividing by the conjugate, we get

$$x^2((30-x)^2+10^2) - (30-x)^2(20^2+x^2) = 0,$$

and simplifying we obtain

$$x^2 - 80x + 1200 = 0,$$

whose roots are $x = 60$ and $x = 20$. Because $x = 60$ does not make sense in this problem, the position of the cable that minimizes its length is $x = 20$ meters. ■

Problem 2. [3 points] Consider the function

$$f(x) = \begin{cases} |x|, & \text{if } |x| < 1, \\ \frac{1}{1+(x+1)^2}, & \text{if } |x| \geq 1 \end{cases}$$

a) Find the sets where it is continuous and differentiable. Find its relative maxima and minima, and the sets where it is convex. **[1.5 points]**

b) Draw a rough sketch of the function using the information obtained above. **[0.5 points]**

c) Find the second degree Taylor polynomial of the function centered at $x = 2$. **[1 point]**

Solution. a) $|x|$ is differentiable everywhere except at $x = 0$. $1/((x+1)^2+1)$ is differentiable everywhere (the denominator is never zero). We need to check what happens at $x = \pm 1$.

At $x = 1$:

$$\lim_{x\to 1^+} \frac{1}{(x+1)^2+1} = \frac{1}{5} \neq 1 = \lim_{x\to 1^-} |x|,$$

so the function is not continuous at $x = 1$.

At $x = -1$:

$$\lim_{x\to -1^-} \frac{1}{(x+1)^2+1} = 1 = 1 = \lim_{x\to -1^+} |x|,$$

so the function is continuous at $x = -1$. Obtaining the derivatives, we get

$$\lim_{x\to -1^+} (|x|)' = -1 \neq 0 = \lim_{x\to -1^-} \frac{-2(x+1)}{((x+1)^2+1)^2},$$

so f is not differentiable at $x = -1$.

There is only one relative minimum, at $x = 0$, and one relative maximum, at $x = -1$.

Regarding convexity: for $|x| < 1$ the second derivative is zero, so f is convex in this interval (not strictly convex, though). Outside $(-1, 1)$, the second derivative of f is

$$f''(x) = \frac{6x^2+12x+4}{((x+1)^2+1)^3},$$

whose roots are $x = -1 \pm \sqrt{3}/3$. The second derivative is negative in $(-1-\sqrt{3}/3, -1+\sqrt{3}/3)$ and nonnegative everywhere else. As a result, f will be convex in $(-\infty, -1-\sqrt{3}/3)$ and in $(1, \infty)$.
c) We have, from part a), that $f(2) = 1/10$, $f'(2) = -6/100$, $f''(2) = 52/1000$, so thet Taylor polynomial of second degree centered at $x = 2$ is

$$P_{2,2} = \frac{1}{10} - \frac{3}{50}(x-2) + \frac{13}{500}(x-2)^2.$$

■

Problem 3. [3 points]

a) Approximate the value of $\log 0.9$ using the third degree Taylor expansion of an appropriate function. **[1.5 points]**

b) What is the (approximate) maximum value of the error made in this approximation? **[1 point]**

c) What should be the degree of the approximation in order for the error to be smaller than 10^{-5}? **[0.5 points]**

Solution. a) The third degree Taylor polynomial of $f(x) = \log(1-x)$ is

$$P_{3,0}(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

which, evaluated at $x = 0.1$, yields

$$P_{3,0}(x) = -\frac{1}{10} - \frac{1}{200} - \frac{1}{3000} \approx -0.1053333\ldots$$

b) The remainder of the previous polynomial is given by

$$R_{3,0}(x) = \left|\frac{f^{(iv)}(c)}{24}x^4\right|$$

where $c \in (0, x)$. The fourth derivative of f is

$$f^(iv)(x) = -\frac{6}{(1-x)^4}$$

whose absolute value is an increasing function in $(0, 0.1)$ and therefore has a maximum value at $x = 0.1$. As a result,

$$R_{3,0}(0.1) = \left|\frac{f^{(iv)}(c)}{24}10^{-4}\right| < \left|\frac{f^{(iv)}(0.1)}{24}10^{-4}\right| = \frac{1}{4 \cdot 0.9^4 \cdot 10^4} = \frac{1}{4 \cdot 6561} \approx 3.8 \cdot 10^{-5}$$

. The actual error made is actually $2.7 \cdot 10^{-5}$.
c) The next remainder is

$$R_{4,0}(0.1) < \left|\frac{f^{(v)}(0.1)}{120}10^{-5}\right| = \frac{1}{5 \cdot 0.9^5 \cdot 10^5} = \frac{1}{295245} \approx 3.4 \cdot 10^{-6},$$

so that will do. ■

Problem 4. [2 points] Calculate the following limits:
a) [1 point]

$$\lim_{x\to 0} \frac{\sin x \cos x - \tan x}{x^2 \log(1+x)}$$

(HINT: $\tan x = x + x^3/3 + o(x^3)$).

b) [1 point]

$$\lim_{x\to\infty} \frac{\cos(\frac{1}{x}) - 1 - \frac{1}{2x^2}}{\sin(\frac{1}{6x^2})}$$

Solution. a) Using the Taylor expansions of the elementary functions in this expression up to third degree, we obtain

$$\lim_{x\to 0} \frac{(x - x^3/6 + o(x^3))(1 - x^2/2 + o(x^3)) - (x + x^3/3 + o(x^3))}{x^3 + o(x^3)} = \lim_{x\to 0} \frac{-x^3 + o(x^3)}{x^3 + o(x^3)} = -1$$

b) First we change the variable $t = 1/x$, then

$$\lim_{t\to 0} \frac{\cos t - 1 - t^2/2}{\sin(t^2/6)} = \lim_{t\to 0} \frac{1 - t^2/2 + o(t^2) - 1 - t^2/2}{t^2/6 + o(t^2} = -\frac{1}{6}$$

■