uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

CALCULUS I - OpenCourseWare

Units 9-11 Exam.

Problem 1. [4 points] Calculate the following primitives:

a) **[2 points]**

$$\int \frac{1}{(x+2)^2(x+3)^2}\,dx$$

b) **[2 points]**

$$\int \frac{5x^5}{\sqrt{1-x^4}}\,dx \quad \text{[HINT: } x^2 = \sin t\text{]}$$

Problem 2. [2 points] Find a function $f(x)$ and a constant c such that

$$\int_c^x t f(t)\,dt = \sin x - x\cos x - \frac{1}{2}x^2$$

for all $x \in \mathbb{R}$.

Problem 3. [4 points] Given the functions

$$f(x) = \frac{3-x}{2} \quad \text{and} \quad g(x) = -\sqrt{x}$$

a) **[2 points]** Calculate the area of the region delimited by f, g and the y axis.

b) **[2 points]** Calculate the volume of the solid generated when the region in a) revolves around the y axis.

SOLUTIONS

Problem 1. **[4 points]** Calculate the following primitives:

a) **[2 points]**

$$\int \frac{1}{(x+2)^2(x+3)^2}\,dx$$

b) **[2 points]**

$$\int \frac{5x^5}{\sqrt{1-x^4}}\,dx \quad \text{[HINT: } x^2 = \sin t\text{]}$$

Solution. a) We decompose into partial fractions:

$$\frac{1}{(x+2)^2(x+3)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2},$$

where

$$A(x+2)(x+3)^2 + B(x+3)^2 + C(x+2)^2(x+3) + D(x+2)^2 = 1,$$

where this equality must hold for all $x \in \mathbb{R}$. Setting $x = -2$ and $x = -3$ yields $B = 1$ and $D = 1$, respectively. Setting $x = 0$ gives $18A+9+12C+4 = 1$ and with $x = -1$ we get $4A+4+2C+1 = 1$, so we have a system of two linear equations with two unknowns

$$18A + 12C = -12$$
$$4A + 2C = -4$$

whose solution is $A = -2, C = 2$. The primitive is then

$$\int \frac{1}{(x+2)^2(x+3)^2}\,dx = -2\int \frac{1}{x+2}\,dx + \int \frac{1}{(x+2)^2}\,dx + 2\int \frac{1}{x+3}\,dx + \int \frac{1}{(x+3)^2}\,dx$$
$$= -2\log|x+2| - \frac{1}{x+2} + 2\log|x+3| - \frac{1}{x+3} + c.$$

b) If $x^2 = \sin t$, then $1 - x^4 = 1 - \sin^2 t = \cos^2 t$ and $2x\,dx = \cos t\,dt$. The primitive becomes

$$\int \frac{5x^5}{\sqrt{1-x^4}}\,dx = \int \frac{5}{2}\sin^2 t\,dt$$
$$= \frac{5}{4}\int (1-\cos 2t)\,dt$$
$$= \frac{5}{4}t - \frac{5}{8}\sin 2t + c,$$

where we have used the trigonometric identity $2\sin^2 t = 1 - \cos 2t$ (otherwise, the integral can be solved by parts). In order to undo the change of variable, we need an expression for $\sin 2t$ in terms of x. But $\sin 2t = 2\sin t\cos t$, and $\cos t = \sqrt{1-x^4}$, so $\sin 2t = 2x^2\sqrt{1-x^4}$. Finally,

$$\int \frac{5x^5}{\sqrt{1-x^4}}\,dx = \frac{5}{4}\arcsin x^2 - \frac{5}{4}x^2\sqrt{1-x^4} + c.$$

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Problem 2. [2 points] Find a function $f(x)$ and a constant c such that

$$\int_c^x tf(t)\,dt = \sin x - x\cos x - \frac{1}{2}x^2$$

for all $x \in \mathbb{R}$.

Solution. Differentiating both sides of the equation we get

$$xf(x) = \cos x - \cos x + x\sin x - x \implies f(x) = \sin x - 1.$$

Now, at $x = c$, the left-hand side is zero, so

$$\sin c - c\cos c - \frac{1}{2}c^2 = 0 \implies c = 0.$$

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Problem 3. [4 points] Given the functions

$$f(x) = \frac{3-x}{2} \quad \text{and} \quad g(x) = -\sqrt{x}$$

a) **[2 points]** Calculate the area of the region delimited by f, g and the y axis.
b) **[2 points]** Calculate the volume of the solid generated when the region in a) revolves around the y axis.

Solution. a) f and g intersect when

$$\frac{3-x}{2} = -\sqrt{x} \implies x^2 - 6x + 9 = 4x \implies x^2 - 10x + 9 = 0$$

There are two solutions of this quadratic equation, $x_- = 9$ and $x_+ = 1$. However, the latter is not a solution of the original equation $\frac{3-x}{2} = -\sqrt{x}$, because the left-hand side is positive while the right-hand side is negative. Therefore, the only interesection point is at $x = 9$. Between $x = 0$ and $x = 9$, $f(x) > g(x)$. Therefore, the area of the region enclosed by the two functions is

$$\begin{aligned}\int_0^9 |f(x) - g(x)|\,dx &= \int_0^9 \left(\frac{3-x}{2} + \sqrt{x}\right) dx \\ &= -\frac{(3-x)^2}{4} + \frac{2}{3}x^{3/2}\Big|_0^9 \\ &= -\frac{36}{4} + \frac{9}{4} + \frac{2}{3}27 \\ &= \frac{45}{4}\end{aligned}$$

b) The volume of a solid of revolution around the y axis is given by

$$\begin{aligned}2\pi \int_0^9 x\left(\frac{3-x}{2} + \sqrt{x}\right) dx &= 2\pi\left(\frac{3x^2}{4} - \frac{x^3}{6} + \frac{2}{5}(x)^{5/2}\right)\Big|_0^9 \\ &= 2\pi\left(\frac{243}{4} - \frac{729}{6} + \frac{486}{5}\right) \\ &\frac{729}{10}\pi.\end{aligned}$$

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