uc3m | Universidad Carlos III de Madrid

OpenCourseWare

Calculus I

Pablo Catalán Fernández y José A. Cuesta Ruiz

Unit 10. Fundamental Theorem of Calculus

Exercises

Problems

Problem 10.1 Find a continuous function f such that $f(0)=0$ and

$$f'(x)=\begin{cases}\dfrac{4-x^2}{(4+x^2)^2}, & x<0,\\ e^{\sqrt{x}}, & x>0.\end{cases}$$

Problem 10.2

(a) Prove that if f is odd then $\displaystyle\int_{-a}^{a} f(x)\,dx=0$.

(b) Prove that if f is even then $\displaystyle\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$.

(c) Calculate the integral $\displaystyle\int_6^{10}\sin\left(\sin\left((x-8)^3\right)\right)dx$.

Problem 10.3 Calculate the following limits:

(i) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$; (ii) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt[n]{e^{2k}}$; (iii) $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n^2-k^2}}$.

Problem 10.4 Calculate $F(x)=\displaystyle\int_{-1}^{x} f(t)\,dt$, with $-1\leqslant x\leqslant 1$, for the following functions:

(i) $f(x)=|x|e^{-|x|}$;

(ii) $f(x)=|x-1/2|$;

(iii) $f(x)=\begin{cases}-1, & -1\leqslant x<0,\\ 1, & 0\leqslant x\leqslant 1;\end{cases}$

(iv) $f(x)=\begin{cases}x^2, & -1\leqslant x<0,\\ x^2-1, & 0\leqslant x\leqslant 1;\end{cases}$

(v) $f(x)=\begin{cases}1, & -1\leqslant x\leqslant 0,\\ x+1, & 0<x\leqslant 1;\end{cases}$

(vi) $f(x)=\begin{cases}1+x, & -1\leqslant x\leqslant -\frac{1}{2},\\ \frac{1}{2}, & -\frac{1}{2}<x<\frac{1}{2},\\ 1-x, & \frac{1}{2}\leqslant x\leqslant 1;\end{cases}$

(vii) $f(x)=\max\left\{\sin(\pi x/2),\cos(\pi x/2)\right\}$.

Problem 10.5 Calculate the following integrals:

(i) $\displaystyle\int_0^{\log 2}\sqrt{e^x-1}\,dx$; (ii) $\displaystyle\int_1^2\frac{\sqrt{x^2-1}}{x}\,dx$.

Problem 10.6 Calculate the derivative of the following functions:

(i) $F(x)=\displaystyle\int_{x^2}^{x^3}\frac{e^t}{t}\,dt$;

(ii) $F(x)=\displaystyle\int_{-x^3}^{x^3}\frac{dt}{1+\sin^2 t}$;

(iii) $F(x)=\displaystyle\int_3^{\int_1^x \sin^3 t\,dt}\frac{dt}{1+\sin^6 t+t^2}$;

(iv) $F(x)=\displaystyle\int_2^{\exp\left\{\int_1^{x^2}\tan\sqrt{t}\,dt\right\}}\frac{ds}{\log s}$;

(v) $F(x)=\displaystyle\int_0^x x^2 f(t)\,dt$, with f continuous in $\mathbb{R}$;

(vi) $F(x)=\sin\left(\displaystyle\int_0^x\sin\left(\int_0^y\sin^3 t\,dt\right)dy\right)$.

Problem 10.7 Find the absolute maximum and minimum in the interval $[1,\infty)$ of the function

$$f(x)=\int_0^{x-1}\left(e^{-t^2}-e^{-2t}\right)dt.$$

HINT: $\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt=\sqrt{\pi}/2$.

Problem 10.8 Prove that the equation

$$\int_0^x e^{t^2}\,dt = 1$$

has a unique solution in $\mathbb{R}$ and that it can be found in the interval $(0,1)$.

Problem 10.9 Let $f(x)$ be a continuous function such that $f(x) > 0$ for all $0 \leqslant x \leqslant 1$, and consider the function

$$F(x) = 2\int_0^x f(t)\,dt - \int_x^1 f(t)\,dt.$$

Determine how many solutions the equation $F(x) = 0$ has in $[0,1]$.

Problem 10.10 Find and classify the local extrema within $(0,\infty)$ of the function

$$G(x) = \int_0^{x^2} \sin t\, e^{\sin t}\,dt.$$

Problem 10.11 Write the equation of the straight tangent to the curve

$$y = \int_{x^2}^{\sqrt{\pi}/2} \tan(t^2)\,dt$$

at the point $x = \sqrt[4]{\pi/4}$.

Problem 10.12 Given the function

$$f(x) = \begin{cases} \dfrac{e^x - 1 - x}{x^2}, & x < 0, \\[2ex] a + b\displaystyle\int_0^x e^{-t^4}\,dt, & x \geqslant 0, \end{cases}$$

calculate a and b so that it is continuous and differentiable.

Problem 10.13 Calculate the following limits:

(i) $\displaystyle\lim_{x\to 0}\frac{1}{x^3}\left(\int_0^x e^{t^2}\,dt - x\right)$; (ii) $\displaystyle\lim_{x\to 0}\frac{\cos x}{x^4}\int_0^x \sin(t^3)\,dt.$

Problem 10.14 Calculate the two one-sided limits at $x = 0$ of the function

$$f(x) = \frac{1}{2x^3}\int_0^{x^2} \tan\sqrt{t}\,dt.$$

Problem 10.15 Consider the function $f(x) = \displaystyle\int_0^{x^2}\frac{\sin t}{t}\,dt$.

(a) Using the Taylor series of $\sin t$ in powers of t, find that of f in powers of x.

(b) Calculate $\displaystyle\lim_{x\to 0}\frac{f(x)}{1-\cos x}$.

(c) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} f(1/n)$.

Problem 10.16 Let $f(x) = \displaystyle\int_{-1/x}^{x}\frac{dt}{a^2+t^2}$. Determine, without computing the integral, for which values of a the function f is constant.

Problem 10.17 Consider the functions $f(x) = e^{x^2} - x^2 - 1$ and $g(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) Write the Taylor series of g in powers of x.
(b) Determine if g has a maximum, a minimum, or an inflection point at $x = 0$.

Problem 10.18
(a) Use the change of variable $t = \sin^2\theta$ to calculate the integral

$$\int_0^1 \arcsin\sqrt{t}\,dt.$$

(b) Consider the function

$$f(x) = \int_0^{\sin^2 x} \arcsin\sqrt{t}\,dt + \int_0^{\cos^2 x} \arccos\sqrt{t}\,dt.$$

Prove that $f(x) = c$, a constant, in the interval $[0, \pi/2]$.
(c) Determine the value of the constant c.

Problem 10.19 The equation

$$\int_0^{g(x)} \left(e^{t^2} + e^{-t^2}\right) dt = x^3 + 3\arctan x$$

defines an injective, differentiable function g in $\mathbb{R}$. Calculate:
(a) $g(0)$, $g'(0)$, and $\left(g^{-1}\right)'(0)$.
(b) $\lim\limits_{x\to 0} \dfrac{g^{-1}(x)}{g(x)}$.

Problem 10.20 Let $f : [-1, 1] \mapsto \mathbb{R}$ be any integrable function.
(a) Prove that

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$$

HINT: Do the change of variables $y = \pi - x$.
(b) Calculate the integral

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$$

Problem 10.21 Let f be a differentiable function such that

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c.$$

Find $f(x)$ and the constant c.

Problem 10.22 Prove that

$$\int_0^x e^{t^2}\,dt \sim \frac{e^{x^2}}{2x} \quad (x \to \infty).$$

Problem 10.23 Let f be a function $n+1$ times differentiable in an interval I, and let $a, x \in I$. Assume that the integral defining the function

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt, \qquad n = 0, 1, \ldots$$

exists.
(a) Calculate $R_0(x)$.
(b) Integrating by parts, find a recurrence formula for $R_n(x)$.
(c) Solve the recurrence and interpret the result.