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Calculus I

Pablo Catalán Fernández y José A. Cuesta Ruiz

Unit 10. Fundamental Theorem of Calculus

Solutions

D.10 Fundamental Theorem of Calculus

Problem 10.1 For $x<0$,

$$\frac{4-x^2}{(4+x^2)^2}=\frac{4+x^2-2x^2}{(4+x^2)^2}=\frac{1}{4+x^2}-\frac{2x^2}{(4+x^2)^2}.$$

Now,

$$\int\frac{dx}{4+x^2}=\frac{1}{2}\arctan\frac{x}{2}$$

and

$$\int\frac{2x^2}{(4+x^2)^2}\,dx=\int x\frac{2x}{(4+x^2)^2}\,dx=-\frac{x}{4+x^2}+\int\frac{dx}{4+x^2}=-\frac{x}{4+x^2}+\frac{1}{2}\arctan\frac{x}{2}.$$

Thus,

$$f(x)=\frac{x}{4+x^2}+a.$$

For $x>0$, with the change $t=\sqrt{x}$ ($dx=2t\,dt$),

$$f(x)=\int e^{\sqrt{x}}\,dx=2\int te^t\,dt=2(t-1)e^t+b=2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+b.$$

Continuity and $f(0)=0$ requires $f(0^-)=a=0$ and $f(0^+)=-2+b=0$, thus

$$f(x)=\begin{cases}\dfrac{x}{4+x^2}, & x<0,\\ 2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+2, & x\geqslant 0.\end{cases}$$

Problem 10.2

(a) Changing $x=-t$,

$$I=\int_{-a}^{a}f(x)\,dx=\int_{-a}^{a}f(-t)\,dt=-\int_{-a}^{a}f(t)\,dt=-I\quad\Rightarrow\quad 2I=0\quad\Rightarrow\quad I=0.$$

(b) Using the same change,

$$\int_{-a}^{a}f(x)\,dx=\int_{0}^{a}f(x)\,dx+\int_{-a}^{0}f(x)\,dx=\int_{0}^{a}f(x)\,dx+\int_{0}^{a}\underbrace{f(-t)}_{=f(t)}\,dt=2\int_{0}^{a}f(x)\,dx.$$

(c) Changing $t=x-8$,

$$\int_{6}^{10}\sin\left(\sin\left((x-8)^3\right)\right)dx=\int_{-2}^{2}\sin\left(\sin\left(t^3\right)\right)dt=0$$

because the integrand is an odd function.

Problem 10.3 These are all Riemann's sums:

(i)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+(k/n)^2}=\int_0^1\frac{dx}{1+x^2}=\arctan 1=\frac{\pi}{4}.$$

(ii)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt[n]{e^{2k}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot e^{2k/n}=\int_0^1 e^{2x}\,dx=\frac{e^2-1}{2}.$$

(iii)

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{\sqrt{n^2-k^2}}=\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{n}\cdot\frac{1}{\sqrt{1-(k/n)^2}}=\int_0^1\frac{dx}{\sqrt{1-x^2}}=\arcsin 1=\frac{\pi}{2}.$$

Problem 10.4

(i) For $x<0$,

$$F(x)=\int_{-1}^{x}(-t)e^t\,dt=(1-x)e^x-\frac{2}{e}.$$

For $x\geqslant 0$,

$$F(x)=\int_{-1}^{0}(-t)e^t\,dt+\int_0^x te^{-t}\,dt=1-\frac{2}{e}+1-(1+x)e^{-x}=2-\frac{2}{e}-(1+x)e^{-x}.$$

(ii) For $x<1/2$,

$$F(x)=\int_{-1}^{x}\left(\frac{1}{2}-t\right)\,dt=\frac{2+x-x^2}{2}=\frac{(2-x)(1+x)}{2}$$

For $x\geqslant 1/2$,

$$F(x)=\int_{-1}^{1/2}\left(\frac{1}{2}-t\right)\,dt+\int_{1/2}^{x}\left(t-\frac{1}{2}\right)\,dt=\frac{9}{4}+\frac{(x-2)(1+x)}{2}.$$

(iii) For $x<0$,

$$F(x)=\int_{-1}^{x}(-1)\,dt=-1-x.$$

For $x\geqslant 0$,

$$F(x)=\int_{-1}^{0}(-1)\,dt+\int_0^x dt=-1+x.$$

Thus, $F(x)=|x|-1$.

(iv) For $x<0$,

$$F(x)=\int_{-1}^{x}t^2\,dt=\frac{x^3+1}{3}.$$

For $x\geqslant 0$,

$$F(x)=\int_{-1}^{0}t^2\,dt+\int_0^x(t^2-1)dt=\int_{-1}^{x}t^2\,dt-\int_0^x dt=\frac{x^3+1}{3}-x=\frac{x^3-3x+1}{3}.$$

(v) For $x\leqslant 0$,

$$F(x)=\int_{-1}^{x}dt=x+1.$$

For $x>0$,

$$F(x)=\int_{-1}^{0}dt+\int_0^x(t+1)dt=\int_{-1}^{x}dt+\int_0^x t\,dt=\frac{x^2}{2}+x+1.$$

(vi) For $x \leqslant -1/2$,

$$F(x) = \int_{-1}^{x} (1+t)\,dt = \frac{(1+x)^2}{2}.$$

For $-1/2 < x < 1/2$,

$$F(x) = \int_{-1}^{-1/2} (1+t)\,dt + \frac{1}{2}\int_{-1/2}^{x} dt = \frac{1}{8} + \frac{2x+1}{4} = \frac{4x+3}{8}.$$

For $x \geqslant 1/2$,

$$F(x) = \int_{-1}^{-1/2} (1+t)\,dt + \frac{1}{2}\int_{-1/2}^{1/2} dt + \int_{1/2}^{x} (1-t)\,dt = \frac{3}{4} - \frac{(1-x)^2}{2}.$$

(vii) For $-1 \leqslant x < 1/2$ we have $\cos(\pi x/2) > \sin(\pi x/2)$, hence

$$F(x) = \int_{-1}^{x} \cos(\pi t/2)\,dt = \frac{2}{\pi}\big[1+\sin(\pi x/2)\big].$$

For $1/2 < x \leqslant 1$ we have $\sin(\pi x/2) > \cos(\pi x/2)$, hence

$$F(x) = \int_{-1}^{1/2} \cos(\pi t/2)\,dt + \int_{1/2}^{x} \sin(\pi t/2)\,dt = \frac{2}{\pi}\left[1+\sqrt{2}-\cos(\pi x/2)\right].$$

Problem 10.5

(i) With the change $t = \sqrt{e^x - 1}$, i.e., $x = \log\left(1+t^2\right)$ (hence $dx = 2t\,dt/(1+t^2)$), we get

$$\int_0^{\log 2} \sqrt{e^x - 1}\,dx = \int_0^1 \frac{2t^2}{1+t^2}\,dt = 2 - 2\arctan 1 = \frac{4-\pi}{2}.$$

(ii) With the change $x = \sec t$ (hence $x^2 - 1 = \tan^2 t$ and $dx = \sec t \tan t\,dt$) we obtain

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \tan^2 t\,dt = (\tan t - t)\Big|_0^{\pi/3} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}.$$

Problem 10.6

(i) $F'(x) = \dfrac{3e^{x^3} - 2e^{x^2}}{x}.$

(ii) $F'(x) = \dfrac{6x^2}{1+\sin^2(x^3)}.$

(iii) $F'(x) = \dfrac{\sin^3 x}{1+\sin^6\left(\int_1^x \sin^3 t\,dt\right) + \left(\int_1^x \sin^3 t\,dt\right)^2}.$

(iv) $F'(x) = \dfrac{2x\tan x}{\displaystyle\int_1^{x^2} \tan\sqrt{t}\,dt}\exp\left\{\displaystyle\int_1^{x^2} \tan\sqrt{t}\,dt\right\}.$

(v) $F'(x) = 2x\displaystyle\int_0^x f(t)\,dt + x^2 f(x).$

(vi) $F'(x) = \cos\left(\displaystyle\int_0^x \sin\left(\int_0^y \sin^3 t\,dt\right)dy\right)\sin\left(\displaystyle\int_0^x \sin^3 t\,dt\right).$

Problem 10.7 $f'(x) = e^{-(x-1)^2} - e^{-2(x-1)}$, so $f'(x) = 0$ when $(x-1)^2 = 2(x-1)$, i.e., when $x = 1$ or $x = 3$. Between those two values $(x-1)^2 < 2(x-1)$, and for $x > 3$ the opposite holds. Therefore $f'(x) > 0$ for $1 < x < 3$ and $f'(x) < 0$ for $x > 3$. Thus there is a local maximum at $x = 3$ —which is the absolute maximum. To obtain the absolute minimum we need to obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\int_0^{x-1} e^{-t^2}\,dt - \int_0^{x-1} e^{-2t}\,dt\right) = \frac{\sqrt{\pi}}{2} - \lim_{x\to\infty}\frac{1}{2}\left(1 - e^{-2(x-1)}\right) = \frac{\sqrt{\pi}-1}{2} > 0.$$

Since $f(1) = 0$, the absolute minimum is reached at $x = 1$.

Problem 10.8 Function $f(x) = \int_0^x e^{t^2}\,dt - 1$ is an increasing function because $f'(x) = e^{x^2} > 0$. Further $f(0) = -1$. On the other hand, $e^{t^2} > 1$ for all $t > 0$, so

$$f(1) = \int_0^1 e^{t^2}\,dt - 1 > \int_0^1 dt - 1 = 0.$$

Therefore $f(x) = 0$ has a unique solution in $(0,1)$.

Problem 10.9 $F(x)$ is a continuous function (is the difference of two integrals) in $[0,1]$. On the other hand,

$$F(0) = 2\cancel{\int_0^0 f(t)\,dt} - \int_0^1 f(t)\,dt = -\int_0^1 f(t)\,dt < 0$$

(it is negative because $f(x) > 0$ in $[0,1]$, therefore the integral is positive), and

$$F(1) = 2\int_0^1 f(t)\,dt - \cancel{\int_1^1 f(t)\,dt} = 2\int_0^1 f(t)\,dt > 0$$

(it is positive for the same reason). Since $F(x)$ has opposite signs at the extremes of the interval it must be zero somewhere in between. Thus, the equation $F(x) = 0$ has at least one solution. To see that there are no more solutions we differentiate

$$F'(x) = 2f(x) - f(x)(-1) = 3f(x) > 0.$$

Therefore $F(x)$ increases monotonically in $[0,1]$, hence can be zero only once in the interval.

Problem 10.10 If $x > 0$ the equation $G'(x) = 2x\sin\left(x^2\right)e^{\sin\left(x^2\right)} = 0$ has solutions $x = \sqrt{n\pi}$, with $n \in \mathbb{N}$. Since the exponential is always positive, the sign of $G'(x)$ is determined by $\sin\left(x^2\right)$. So it starts being positive and alternates sign every other solution. So $\sqrt{(2k-1)\pi}$ are maxima and $\sqrt{2k\pi}$ are minima ($k \in \mathbb{N}$).

Problem 10.11 For $x = \sqrt[4]{\pi/4}$ we get $y = 0$. On the other hand, since $y' = -2x\tan\left(x^4\right)$, the slope at $x = \sqrt[4]{\pi/4}$ will be $-2\sqrt[4]{\pi/4} = -\sqrt[4]{4\pi}$. This yields for the tangent straight line the equation

$$y = -\sqrt[4]{4\pi}\left(x - \sqrt[4]{\pi/4}\right) = \sqrt{\pi} - \sqrt[4]{4\pi}x.$$

Problem 10.12 If the function must be continuous at 0 then $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x)$. But

$$\begin{aligned}
\lim_{x\to0^-} f(x) &= \lim_{x\to0}\frac{e^x - 1 - x}{x^2} = \lim_{x\to0}\frac{1 + x + x^2/2 + o(x^2) - 1 - x}{x^2} = \lim_{x\to0}\frac{x^2/2 + o(x^2)}{x^2} \\
&= \lim_{x\to0}\left[\frac{1}{2} + o(1)\right] = \frac{1}{2}, \\
\lim_{x\to0^+} f(x) &= \lim_{x\to0}\left(a + b\int_0^x e^{-t^4}\,dt\right) = a + b\int_0^0 e^{-t^4}\,dt = a.
\end{aligned}$$

Hence $a = 1/2$. Now, for the function to be differentiable at $x = 0$ it must hold

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x}.$$

Since $f(0) = 1/2$,

$$\begin{aligned}
\lim_{x\to 0^-} \frac{f(x) - f(0)}{x} &= \lim_{x\to 0} \frac{\dfrac{e^x - 1 - x}{x^2} - \dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3} \\
&= \lim_{x\to 0} \frac{1 + x + x^2/2 + x^3/6 + o(x^3) - 1 - x - x^2/2}{x^3} = \lim_{x\to 0} \frac{x^3/6 + o(x^3)}{x^3} \\
&= \lim_{x\to 0} \left[\frac{1}{6} + o(1)\right] = \frac{1}{6}, \\
\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} &= \lim_{x\to 0} \frac{\dfrac{1}{2} + b\displaystyle\int_0^x e^{-t^4}\,dt - \dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{b}{x} \int_0^x e^{-t^4}\,dt = b\frac{d}{dx}\left(\int_0^x e^{-t^4}\,dt\right)\Bigg|_{x=0} \\
&= be^{-x^4}\Big|_{x=0} = b.
\end{aligned}$$

Therefore $b = 1/6$.

Here is a shorter alternative. We can Taylor expand both functions up to first order. On the one hand

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + 0(x^3),$$

therefore

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + o(x).$$

On the other hand if

$$g(x) = \int_0^x e^{-t^4}\,dt$$

then $g(0) = 0$, $g'(x) = e^{-x^4}$ and $g'(0) = 1$, so

$$g(x) = x + o(x),$$

therefore

$$a + b\int_0^x e^{-t^4}\,dt = a + bx + o(x).$$

If $f(x)$ has to be continuous and differentiable at $x = 0$ both expansions must coincide up to first order, hence we obtain the same values for a and b.

Problem 10.13

(i) Using l'Hôpital's rule once we get

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{3x^2} = \frac{1}{3}.$$

(ii) Since $\lim_{x\to 0}\cos x = 1$,

$$\lim_{x\to 0}\frac{\cos x}{x^4}\int_0^x \sin\left(t^3\right)\,dt = \lim_{x\to 0}\frac{1}{x^4}\int_0^x \sin\left(t^3\right)\,dt.$$

Applying l'Hôpital's rule once we get

$$\lim_{x\to 0}\frac{\sin\left(x^3\right)}{4x^3} = \frac{1}{4}.$$

Problem 10.14 Using l'Hôpital's rule once we get

$$\lim_{x\to 0^{\pm}}\frac{2x\tan|x|}{6x^2} = \lim_{x\to 0^{\pm}}\frac{\tan|x|}{3x} = \pm\frac{1}{3}.$$

Problem 10.15

(a) Since

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}, \qquad \int_0^{x^2} t^{2n}\,dt = \frac{x^{4n+2}}{2n+1},$$

we obtain

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{x^{2n+2}}{2n+1}.$$

(b) $1-\cos x = \frac{x^2}{2} + o\left(x^2\right)$ $(x\to 0)$ and $f(x) = x^2 + o\left(x^2\right)$ $(x\to 0)$, so

$$\lim_{x\to 0}\frac{f(x)}{1-\cos x} = 2.$$

(c) The series converges because

$$f(1/n) = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Problem 10.16

$$f'(x) = \frac{1}{a^2+x^2} - \frac{1}{x^2}\frac{1}{a^2+1/x^2} = \frac{1}{a^2+x^2} - \frac{1}{a^2x^2+1},$$

so in order to have $f'(x) = 0$ for any x we need $a = \pm 1$.

Problem 10.17

(a) $f(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!} - x^2 - 1 = \sum_{n=2}^{\infty}\frac{x^{2n}}{n!}$. Then

$$g(x) = \sum_{n=2}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}.$$

(b) Since $g(x) = x^5/10 + o\left(x^5\right)$ $(x\to 0)$ —i.e., the first nonzero derivative at $x = 0$ is the fifth—, the point $x = 0$ is an inflection point.

Problem 10.18

(a) $dt = 2\sin\theta\cos\theta\,d\theta = \sin 2\theta\,d\theta$, therefore

$$I = \int_0^1 \arcsin\sqrt{t}\,dt = \int_0^{\pi/2} \arcsin(\sin\theta)\,\sin 2\theta\,d\theta = \int_0^{\pi/2} \theta\sin 2\theta\,d\theta.$$

We can now integrate by parts, where $u = \theta$ and $v' = \sin 2\theta$, and then

$$I = -\frac{\theta}{2}\cos 2\theta\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\,d\theta = \frac{\pi}{4} + \frac{1}{4}\sin 2\theta\Big|_0^{\pi/2} = \frac{\pi}{4} + 0.$$

Thus

$$\int_0^1 \arcsin\sqrt{t}\,dt = \frac{\pi}{4}.$$

(b) Differentiating,

$$f'(x) = 2\sin x\cos x\arcsin(\sin x) - 2\cos x\sin x\arccos(\cos x) = x\sin 2x - x\sin 2x = 0.$$

Therefore $f(x)$ is constant.

(c) We can calculate c by substituting any value of x, for instance $x = \pi/2$. Then

$$c = f(\pi/2) = \int_0^1 \arcsin\sqrt{t}\,dt + \int_0^0 \arccos\sqrt{t}\,dt = \int_0^1 \arcsin\sqrt{t}\,dt.$$

But this is precisely the integral we have obtained in (a), so $c = \pi/4$.

Problem 10.19

(a) Setting $x = 0$ in the equation

$$\int_0^{g(0)} \left(e^{t^2} + e^{-t^2}\right) dt = 0.$$

Since the integrand is a strictly positive function, the only possibility for this equation to hold is that $g(0) = 0$.

Differentiating,

$$g'(x)\left(e^{g(x)^2} + e^{-g(x)^2}\right) = 3x^2 + \frac{3}{1+x^2},$$

thus, using $g(0) = 0$, we obtain $g'(0) = 3/2$.

Finally, we know that $g(0) = 0$ so $g^{-1}(0) = 0$. Then

$$\left(g^{-1}\right)'(0) = \frac{1}{g'\left(g^{-1}(0)\right)} = \frac{1}{g'(0)} = \frac{2}{3}.$$

(b) Since it is an indeterminacy $\frac{0}{0}$ we can use l'Hôpital's rule and calculate

$$\lim_{x\to 0}\frac{\left(g^{-1}\right)'(x)}{g'(x)} = \frac{\left(g^{-1}\right)'(0)}{g'(0)} = \frac{2/3}{3/2} = \frac{4}{9}.$$

Problem 10.20

(a) With this change of variables the limits remain the same, so

$$I = \int_0^{\pi} x f(\sin x)\, dx = \int_0^{\pi} (\pi - y) f\big(\sin(\pi - y)\big)\, dy.$$

But since $\sin(\pi - y) = \sin y$, we have

$$I = \int_0^{\pi} (\pi - y) f(\sin y)\, dy = \pi \int_0^{\pi} f(\sin y)\, dy - I.$$

Thus

$$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx.$$

(b) Since

$$\frac{\sin x}{1+\cos^2 x} = \frac{\sin x}{2-\sin^2 x} = f(\sin x),$$

we are in the situation described in the previous item. Hence

$$\begin{aligned} I &= \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = -\frac{\pi}{2} \int_0^{\pi} \frac{(\cos x)'}{1+\cos^2 x}\, dx \\ &= -\frac{\pi}{2} \arctan(\cos x)\Big|_0^{\pi} = -\frac{\pi}{2}(-2\arctan 1) = \frac{\pi^2}{4}. \end{aligned}$$

Problem 10.21 Differentiating the equation,

$$f(x) = -x^2 f(x) + 2x^{15} + 2x^{17} \quad\Rightarrow\quad (1+x^2) f(x) = 2x^{15}(1+x^2) \quad\Rightarrow\quad f(x) = 2x^{15}.$$

Now substituting back into the equation and setting $x = 1$,

$$\int_0^1 f(t)\, dt = \frac{1}{8} + \frac{1}{9} + c \quad\Rightarrow\quad \frac{t^{16}}{8}\Big|_0^1 = \frac{1}{8} + \frac{1}{9} + c \quad\Rightarrow\quad \frac{1}{8} = \frac{1}{8} + \frac{1}{9} + c \quad\Rightarrow\quad c = -\frac{1}{9}.$$

Problem 10.22 By definition $f(x) \sim g(x)$ $(x \to a)$ if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

In our case we have to calculate the limit

$$\ell = \lim_{x\to\infty} \frac{\int_0^x e^{t^2}\, dt}{e^{x^2}/2x}.$$

Since this is a $\frac{\infty}{\infty}$ indeterminacy, we can apply l'Hôpital and obtain

$$\ell = \lim_{x\to\infty} \frac{e^{x^2}}{(4x^2 e^{x^2} - 2e^{x^2})/4x^2} = \lim_{x\to\infty} \frac{4x^2}{4x^2-2} = 1.$$

This proves the equivalence.

Problem 10.23

(a) $R_0(x) = \displaystyle\int_a^x f'(t)\, dt = f(x) - f(a).$

(b) $R_n(x) = \displaystyle\frac{1}{n!}\left[(x-t)^n f^{(n)}(t)\Big|_a^x + n\int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt\right] = -\frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n-1}(x).$

(c) Using the recurrence iteratively we obtain

$$
\begin{aligned}
R_0(x) &= f(x) - f(a), \\
R_1(x) &= f(x) - f(a) - f'(a)(x-a), \\
R_2(x) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2, \\
&\vdots \\
R_n(x) &= f(x) - f(a) - f'(a)(x-a) - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n.
\end{aligned}
$$

In other words, $R_n(x) = f(x) - P_{n,a}(x)$, where $P_{n,a}(x)$ is Taylor's polynomial of $f(x)$ at the point a. Function $R_n(x)$ is therefore the remainder of order n of Taylor's approximation.