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Calculus I

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Unit 11. Geometric Applications of Integrals

Exercises

Problems

Problem 11.1 Calculate the area delimited by the following curves:
(i) $y = x^2$, $y = (x-2)^2$, $y = (2-x)/6$;
(ii) $x^2 + y^2 = 1$, $x^2 + y^2 = 2x$;
(iii) $y = \dfrac{1-x}{1+x}$, $y = \dfrac{2-x}{1+x}$, $y = 0$, $y = 1$;
(iv) one loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

Problem 11.2 Determine the area between the curve $f(x) = \dfrac{x(x^2-1)}{(x^2+1)^{3/2}}$ and the X axis.

Problem 11.3 Calculate the area delimited by the following curves:
(i) $r = a\theta$ (Archimedes's spiral), $0 \leqslant \theta \leqslant 2\pi$, and the segment $\{(x,0) : 0 \leqslant x \leqslant 2\pi a\}$;
(ii) a petal of the three-petal rose $r = a\cos 3\theta$, $-\pi/6 \leqslant \theta \leqslant \pi/6$;
(iii) half a *lemniscata* $r = a\sqrt{\cos 2\theta}$, $-\pi/4 \leqslant \theta \leqslant \pi/4$.

Problem 11.4 Let A the plane figure limited by the curves $y = x^2$ and $y = \sqrt{x}$. Determine:
(a) the area of A;
(b) the volume of the solid generated when A revolves around the X axis.

Problem 11.5 Compute the volume of the solids generated when the following sets revolve around the X asis:
(i) $0 \leqslant y \leqslant 1 + \sin x$, $0 \leqslant x \leqslant 2\pi$;
(ii) $R^2 \leqslant x^2 + y^2 \leqslant 4R^2$;
(iii) plane figure delimited by the curves $y = \sin x$ and $y = x$ with $0 \leqslant x \leqslant \pi$.

Problem 11.6 Compute the volume of the following solids:
(i) the solid generated when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ revolves around the X axis;
(ii) same thing around the Y axis;
(iii) the solid whose base is the ellipse above and whose sections perpendicular to the X axis are isosceles triangles of height 2.

Problem 11.7
(a) Calculate the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
(b) Calculate the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.
(c) Check the result of Problem 11.6 (i) and (ii) as particular cases of the previous result.

HINT: Notice that intersecting the ellipsoid by planes parallel to the coordinate planes ($x = 0$, $y = 0$, or $z = 0$) we obtain ellipses.

Problem 11.8 Calculate the length of the following curves:
(i) catenary: $y = e^{x/2} + e^{-x/2}$, $0 \leqslant x \leqslant 2$;
(ii) cycloid: $x(t) = a(t - \sin t)$, $y(t) = a(1 - \cos t)$, $0 \leqslant t \leqslant 2\pi$;
(iii) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4$;
(iv) tractrix: $y = a\log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}$, $a/2 \leqslant x \leqslant a$;
(v) cardioid: $r = 1 + \cos\theta$, $0 \leqslant \theta \leqslant 2\pi$.