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Calculus I

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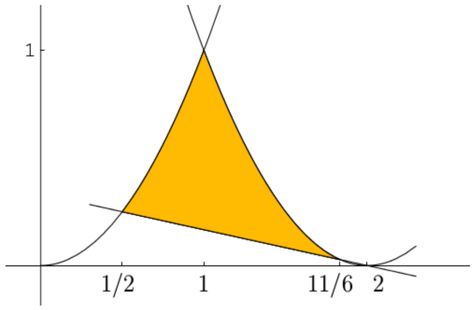
Unit 11. Geometric Applications of Integrals

Solutions

D.11 Geometric Applications of Integrals

Problem 11.1

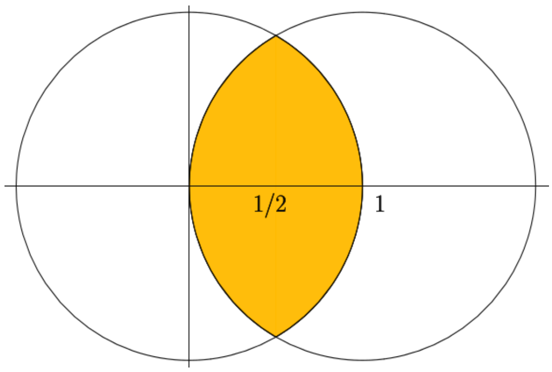
(i) Here is the figure delimited by the three curves:



$y = x^2$ and $y = (x-2)^2$ meet at $x = 1$; $y = x^2$ and $y = (2-x)/6$ meet at $x = 1/2$; and $y = (x-2)^2$ and $y = (2-x)/6$ meet at $x = 11/6$ and $x = 2$. The area enclosed by this three curves is calculated as

$$A = \int_{1/2}^{1} \left(x^2 - \frac{2-x}{6}\right) dx + \int_{1}^{11/6} \left((x-2)^2 - \frac{2-x}{6}\right) dx = \frac{71}{162}.$$

(ii) Here is the figure delimited by the two circumferences:

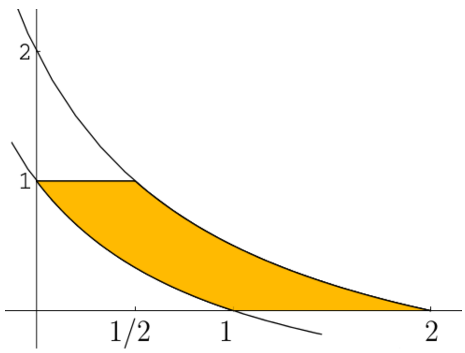


They meet at $x = 1/2$. By symmetry,

$$A = 4\int_{1/2}^{1} \sqrt{1-x^2}\, dx = 4\int_{\pi/6}^{\pi/2} \cos^2 t\, dt = 2\int_{\pi/6}^{\pi/2} (1+\cos 2t)\, dt = \frac{2\pi}{3} - \frac{\sqrt{3}}{2},$$

using the change $x = \sin t$ and the identity $2\cos^2 t = 1 + \cos 2t$.

(iii) Here is the figure delimited by the four curves:



The area is then

$$A = \int_0^{1/2} \left(1 - \frac{1-x}{1+x}\right) dx + \int_{1/2}^{1} \left(\frac{2-x}{1+x} - \frac{1-x}{1+x}\right) dx + \int_1^2 \frac{2-x}{1+x} dx$$
$$= \int_0^{1/2} \frac{2x}{1+x} dx + \int_{1/2}^{1} \frac{dx}{1+x} + \int_1^2 \frac{2-x}{1+x} dx = 1 - 2\log(1+x)\Big|_0^{1/2} + \log(1+x)\Big|_{1/2}^{1}$$
$$+ 3\log(1+x)\Big|_1^2 - 1 = -2\log(3/2) + \log 2 - \log(3/2) + 3\log 3 - 3\log 2 = \log 2.$$

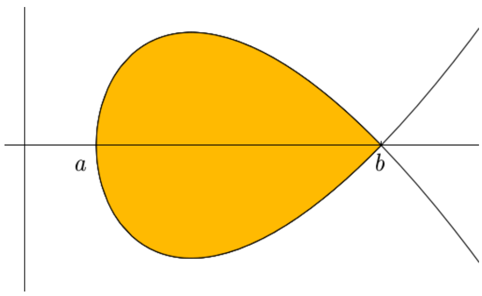
The result is easier to obtain if we express the curves as

$$x = \frac{1-y}{1+y}, \qquad x = \frac{2-y}{1+y},$$

for then

$$A = \int_0^1 \left(\frac{2-y}{1+y} - \frac{1-y}{1+y}\right) dy = \int_0^1 \frac{dy}{1+y} dy = \log 2.$$

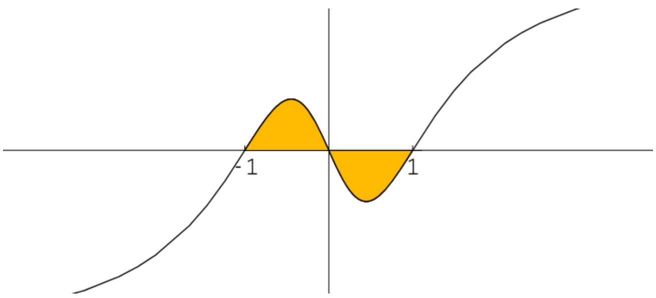
(iv) Here is the figure delimited by the curve:



By symmetry, the area is

$$A = 2\int_a^b (b-x)\sqrt{x-a}\,dx = 2\int_a^b \left[(b-a)-(x-a)\right]\sqrt{x-a}\,dx = \frac{8}{15}(b-a)^{5/2}.$$

Problem 11.2 Given the figure



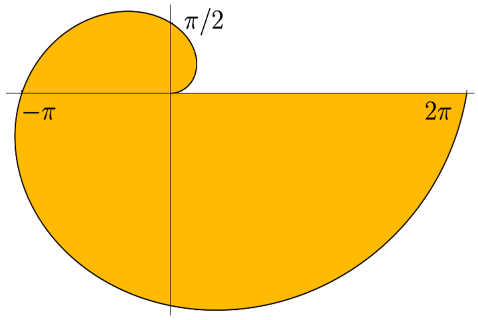
the area is obtained, by symmetry, as

$$A = 2\int_0^1 \frac{x(1-x^2)}{(x^2+1)^{3/2}} dx = 2\int_0^{\pi/4} \tan t \underbrace{(1-\tan^2 t)}_{=2-\frac{1}{\cos^2 t}} \cos^3 t \frac{dt}{\cos^2 t} = 2\int_0^{\pi/4} \left(2\sin t - \frac{\sin t}{\cos^2 t}\right) dt$$
$$= 4(-\cos t)\Big|_0^{\pi/4} - \frac{2}{\cos t}\Big|_0^{\pi/4} = 6 - 4\sqrt{2},$$

where we have made the change of variable $x = \tan t$ and used the identity $1 + \tan^2 x = 1/\cos^2 x$.

Problem 11.3

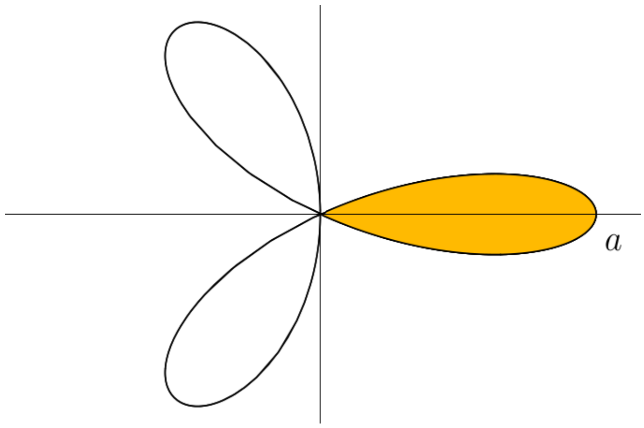
(i) Here is the figure:



The area is obtained as

$$A = \frac{1}{2}\int_0^{2\pi} a^2\theta^2\,d\theta = \frac{4}{3}\pi^3 a^2.$$

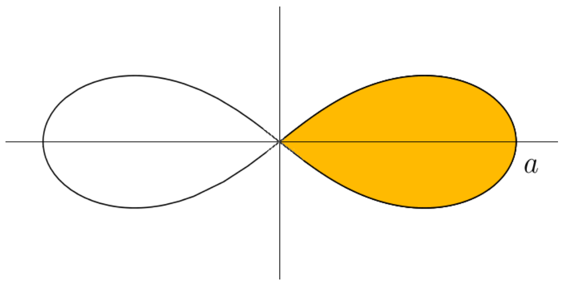
(ii) Here is the figure:



The area is obtained, by symmetry, as

$$A = \int_0^{\pi/6} a^2\cos^2 3\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta = \frac{\pi}{12}a^2.$$

(iii) Here is the figure:



The area is obtained, by symmetry, as

$$A = \int_0^{\pi/4} a^2\cos 2\theta\,d\theta = \frac{a^2}{2}.$$

Problem 11.4

(a) Both curves meet at $x=0$ and $x=1$, and within $[0,1]$ we have $\sqrt{x} \geqslant x^2$. Then

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{1}{3}.$$

(b)

$$V = \pi \int_0^1 x\,dx - \pi \int_0^1 x^4\,dx = \frac{3\pi}{10}.$$

Problem 11.5

(i)

$$\begin{aligned} V =& \pi \int_0^{2\pi} (1+\sin x)^2\,dx = \pi \int_0^{2\pi} \left(1 + 2\sin x + \sin^2 x\right) dx \\ =& \pi \int_0^{2\pi} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\sin 2x\right) dx = 3\pi^2. \end{aligned}$$

(ii)

$$V = \frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3 = \frac{28}{3}\pi R^3.$$

(iii) Since $x \geqslant \sin x$ within $[0,\pi]$,

$$V = \pi \int_0^{\pi} \left(x^2 - \sin^2 x\right) dx = \pi \int_0^{\pi} \left(x^2 - \frac{1}{2} + \frac{1}{2}\sin 2x\right) dx = \frac{\pi^4}{3} - \frac{\pi^2}{2}.$$

Problem 11.6

(i)

$$V = \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \frac{4}{3}\pi b^2 a.$$

(ii)

$$V = 4\pi \int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\,dx = -\frac{4}{3}\pi b a^2 \left(1 - \frac{x^2}{a^2}\right)^{3/2}\Bigg|_0^a = \frac{4}{3}\pi b a^2.$$

(iii) The area of the triangular section at x will be

$$a(x) = 2b\sqrt{1 - \frac{x^2}{a^2}},$$

hence

$$V = 2b\int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}}\,dx = 2ab\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = ab\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta = \pi ab,$$

using the change of variable $x = a\sin\theta$.

Problem 11.7

(a) By symmetry

$$A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx = \pi ab$$

(see Problem 11.6(iii)).

(b) Rewrite the equation of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2} \quad \Rightarrow \quad \frac{x^2}{a(z)^2}+\frac{y^2}{b(z)^2}=1,$$

where

$$a(z)=a\sqrt{1-\frac{z^2}{c^2}}, \qquad b(z)=b\sqrt{1-\frac{z^2}{c^2}}.$$

This means that the sections of the ellipsoid perpendicular to the Z axis are ellipses, with axis $a(z)$ and $b(z)$ $(-c\leqslant z\leqslant c)$. Their area is $A(z)=\pi a(z)b(z)$ (see (a)), therefore

$$V=\pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)dz=\frac{4}{3}\pi abc.$$

(c) If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ rotates around the X axis, it generates an ellipsoid with $c=b$, hence $V=4\pi ab^2/3$. If it does around the Y axis, the ellipsoid will have $c=a$, hence $V=4\pi a^2b/3$.

Problem 11.8

(i) $\mathbf{r}(x)=\left(x,e^{x/2}+e^{-x/2}\right)$, thus $\mathbf{r}'(x)=\left(1,(e^{x/2}-e^{-x/2})/2\right)$ and $\|\mathbf{r}'(x)\|=\sqrt{1+\sinh^2(x/2)}=\cosh(x/2)$. Accordingly

$$L=\int_0^2\cosh\frac{x}{2}\,dx=2\sinh 1=e-e^{-1}.$$

(ii) $\mathbf{r}'(t)=\left(a(1-\cos t),a\sin t\right)$,

$$\|\mathbf{r}'(t)\|=a\sqrt{(1-\cos t)^2+\sin^2 t}=a\sqrt{2(1-\cos t)}=2a\left|\sin\frac{t}{2}\right|$$

Therefore

$$L=2a\int_0^{2\pi}\sin\frac{t}{2}\,dt=4a\int_0^{\pi}|\sin u|\,du=8a.$$

(iii) One arc of the curve can be parametrised $\mathbf{r}(x)=\left(x,(4-x^{2/3})^{3/2}\right)$, where $0\leqslant x\leqslant 8$ (the other three have identical length). Thus $\mathbf{r}'(x)=\left(1,-x^{-1/3}(4-x^{2/3})^{1/2}\right)$ and $\|\mathbf{r}'(x)\|=\sqrt{1+x^{-2/3}(4-x^{2/3})}=2x^{-1/3}$. Accordingly

$$L=8\int_0^8 x^{-1/3}\,dx=48.$$

(iv) Taking $\mathbf{r}(x)=\left(x,y(x)\right)$ we get, after a lengthy calculation,

$$\mathbf{r}'(x)=\left(1,-\frac{\sqrt{a^2-x^2}}{x}\right) \quad \Rightarrow \quad \|\mathbf{r}'(x)\|=\sqrt{1+\frac{a^2-x^2}{x^2}}=\frac{a}{x}.$$

Therefore

$$L=a\int_{a/2}^{a}\frac{dx}{x}=a\log 2.$$

(v) The parametrisation is $\mathbf{r}(\theta) = \big(r(\theta)\cos\theta, r(\theta)\sin\theta\big)$, so

$$\mathbf{r}'(\theta) = \big(r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta\big)$$

and therefore, using $\cos^2\theta + \sin^2\theta = 1$,

$$\|\mathbf{r}'(\theta)\| = \sqrt{r'(\theta)^2 + r(\theta)^2} = \sqrt{\sin^2\theta + (1+\cos\theta)^2} = \sqrt{2(1+\cos\theta)} = 2\left|\cos\frac{\theta}{2}\right|.$$

Accordingly,

$$L = 2\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right| d\theta = 4\int_0^{\pi}|\cos t|\,dt = 8\int_0^{\pi/2}\cos t\,dt = 8.$$