
OpenCourseWare

Calculus I

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Unit 1. The Real Line

Exercises



Problems

Problem 1.1 Given the real numbers $0 < a < b$ and $c > 0$, prove the inequalities

$$(a) \ a < \sqrt{ab} < \frac{a+b}{2} < b, \quad (b) \ \frac{a}{b} < \frac{a+c}{b+c}.$$

Problem 1.2 Prove that $|a+b| = |a| + |b|$ if and only if $ab \geq 0$.

Problem 1.3 Prove that

$$(a) \ \max\{x, y\} = \frac{x+y+|x-y|}{2}, \quad (b) \ \min\{x, y\} = \frac{x+y-|x-y|}{2}.$$

Problem 1.4 Find, using the absolute value, a formula to express the function

$$\varphi(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Problem 1.5 Factor out the following expressions of $n \in \mathbb{N}$, so that the corresponding statements become self-evident:

- (a) $n^2 - n$ is even,
- (b) $n^3 - n$ is a multiple of 6,
- (c) $n^2 - 1$ is a multiple of 8 when n is odd.

Problem 1.6 Prove by induction the following statements valid for all $n \in \mathbb{N}$:

- (a) $a^n - b^n = (a-b) \sum_{k=1}^n a^{n-k} b^{k-1}$ for all $n \in \mathbb{N}$,
- (b) $n^5 - n$ is a multiple of 5 for all $n \in \mathbb{N}$,
- (c) $(1+x)^n \geq 1+nx$ if $x \geq -1$.

Problem 1.7 Prove by induction the following statements valid for all natural numbers $n > 1$:

- (a) $n! < \left(\frac{n+1}{2}\right)^n$,
- (b) $2!4!\cdots(2n)! > [(n+1)!]^n$,
- (c) $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

HINT: In (a) use the inequality $\left(1 + \frac{1}{n+1}\right)^{n+1} > 2$, valid for all $n \in \mathbb{N}$. In (b) prove first that $(2n+2)! > (n+2)^n(n+2)!$.

Problem 1.8

- (a) Show, with an example, that the sum of two irrational numbers can be rational.
- (b) Show, with an example, that the product of two irrational numbers can be rational.
- (c) Is it possible to find irrational numbers x and y such that $x^y \in \mathbb{Q}$?

Problem 1.9 Prove that

- (a) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$,
- (b) $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square (HINT: write $n = k^2 r$, where r does not contain any square factor),
- (c) $\sqrt{n-1} + \sqrt{n+1} \notin \mathbb{Q}$ for all $n \in \mathbb{N}$.

Problem 1.10 Prove the identity, valid for all $x \in \mathbb{R}$,

$$\left(\frac{x+|x|}{2}\right)^2 + \left(\frac{x-|x|}{2}\right)^2 = x^2.$$

Problem 1.11 Identify the following sets:

- (i) $A = \{x \in \mathbb{R} : |x-3| \leq 8\}$,
 (ii) $B = \{x \in \mathbb{R} : 0 < |x-2| < 1/2\}$,
 (iii) $C = \{x \in \mathbb{R} : x^2 - 5x + 6 \geq 0\}$,
 (iv) $D = \{x \in \mathbb{R} : x^3(x+3)(x-5) < 0\}$,
 (v) $E = \left\{x \in \mathbb{R} : \frac{2x+8}{x^2+8x+7} > 0\right\}$,
 (vi) $F = \left\{x \in \mathbb{R} : \frac{4}{x} < x\right\}$,
 (vii) $G = \{x \in \mathbb{R} : 4x < 2x+1 \leq 3x+2\}$,
 (viii) $H = \{x \in \mathbb{R} : |x^2 - 2x| < 1\}$,
 (ix) $I = \{x \in \mathbb{R} : |x-1||x+2| = 10\}$,
 (x) $J = \{x \in \mathbb{R} : |x-1| + |x+2| > 1\}$.

Problem 1.12 Given real numbers $a < b$ we define, for each $t \in \mathbb{R}$, the real number $x(t) = (1-t)a + tb$. Identify the following sets:

- (i) $A = \{x(t) : t = 0, 1, 1/2\}$,
 (ii) $B = \{x(t) : t \in (0, 1)\}$,
 (iii) $C = \{x(t) : t < 0\}$,
 (iv) $D = \{x(t) : t > 1\}$.

Problem 1.13 Find supremum and infimum (deciding whether they are maximum and minimum respectively) of the following sets:

- (i) $A = \{-1\} \cup [2, 3)$,
 (ii) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0, 1]$,
 (iii) $C = \{2 + 1/n : n \in \mathbb{N}\}$,
 (iv) $D = \{(n^2 + 1)/n : n \in \mathbb{N}\}$,
 (v) $E = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$,
 (vi) $F = \{x \in \mathbb{R} : (x-a)(x-b)(x-c)(x-d) < 0\}$,
 with $a < b < c < d$ given real numbers,
 (vii) $G = \{2^{-p} + 5^{-q} : p, q \in \mathbb{N}\}$,
 (viii) $H = \{(-1)^n + 1/m : n, m \in \mathbb{N}\}$,