uc3m | Universidad Carlos III de Madrid

OpenCourseWare

Calculus I

Pablo Catalán Fernández y José A. Cuesta Ruiz

Unit 1. The Real Line

Solutions

D. Solutions to exercises

D.1 The Real Line

Problem 1.1

(a) First of all, if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$ (for suppose it were false and $\sqrt{b} \leqslant \sqrt{a}$; then squaring we would have $b \leqslant a$, which we know is false). Now, multiplying this last inequality by $\sqrt{a} > 0$ we get

$$\left(\sqrt{a}\right)^2 < \sqrt{a}\sqrt{b} \qquad \Leftrightarrow \qquad a < \sqrt{ab}.$$

For the second inequality, we start off from the fact that $\left(\sqrt{a}-\sqrt{b}\right)^2 > 0$ and then expand the binomial to obtain $a - 2\sqrt{ab} + b > 0$. Adding $2\sqrt{ab}$ to the inequality we get $a + b > 2\sqrt{ab}$, and finally multiplying by $1/2$ we obtain

$$\frac{a+b}{2} > \sqrt{ab}.$$

The last inequality is obtained by adding b to $a < b$ to obtain

$$a + b < b + b = 2b \qquad \Leftrightarrow \qquad \frac{a+b}{2} < b.$$

(b) Since $0 < a < b$ and $c > 0$, then $ac < bc$. Now we add ab to the inequality and obtain $ab + ac < ab + bc$. Factoring out the common factor in each side of it,

$$a(b+c) < b(a+c) \qquad \Leftrightarrow \qquad \frac{a}{b} < \frac{a+c}{b+c}$$

because dividing by $a > 0$ or $b > 0$ does not change the inequality.

Problem 1.2 Proving this amounts to proving two statements: (i) that $|a+b| = |a| + |b|$ implies $ab \geqslant 0$, and (ii) that $ab \geqslant 0$ implies $|a+b| = |a| + |b|$.

Let us start with (i). If we square the expression we get $|a+b|^2 = (|a|+|b|)^2$. But $|a+b|^2 = (a+b)^2$. Now expanding both binomials we obtain

$$a^2 + 2ab + b^2 = |a|^2 + 2|ab| + |b|^2,$$

and cancelling common terms in both sides we end up with $2ab = 2|ab| \geqslant 0$.

As for (ii), $ab \geqslant 0$ means that a and b have both the same sign. Suppose $a \geqslant 0$ and $b \geqslant 0$. Then $|a+b| = a+b = |a|+|b|$. Suppose now that $a \leqslant 0$ and $b \leqslant 0$. Then we can write $a = -|a|$ and $b = -|b|$, and therefore

$$|a+b| = \big|-|a|-|b|\big| = \big|-(|a|+|b|)\big| = |a|+|b|.$$

Problem 1.3

(a) Suppose $x \geqslant y$. Then $\max\{x,y\} = x$ and $|x-y| = x-y$, so

$$\frac{x+y+|x-y|}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x.$$

Suppose now that $x < y$. Then $\max\{x,y\} = y$ and $|x-y| = y-x$, so

$$\frac{x+y+|x-y|}{2} = \frac{x+y+y-x}{2} = \frac{2y}{2} = y.$$

In both cases the two sides of the equality yield the same result.

(b) Suppose $x \geqslant y$. Then $\min\{x,y\} = y$ and $|x-y| = x-y$, so

$$\frac{x+y-|x-y|}{2} = \frac{x+y-x+y}{2} = \frac{2y}{2} = y.$$

Suppose now that $x < y$. Then $\max\{x,y\} = x$ and $|x-y| = y-x$, so

$$\frac{x+y-|x-y|}{2} = \frac{x+y-y+x}{2} = \frac{2x}{2} = x.$$

Again, whichever the case, both sides of the equality yield the same result.

Problem 1.4 Clearly $\varphi(x) = \max\{x,0\}$, so using the formulas from the previous exercise

$$\varphi(x) = \frac{x+|x|}{2}.$$

Problem 1.5

(a) $n^2 - n = n(n-1)$, which is even because it is the product of two consecutive numbers —one of which must be even.

(b) $n^3 - n = n(n^2-1) = (n-1)n(n+1)$, hence must be a multiple of 3 because it is the product of three consecutive numbers —one of which must be a multiple of 3—, but is also a multiple of 2 because in every three consequtive numbers at least one is even —and possibly two. Thus it is a multiple of both, 2 and 3, therefore is a multiple of 6.

(c) Odd numbers are written as $n = 2k-1$, with $k \in \mathbb{N}$. Hence $n^2 - 1 = (2k-1)^2 - 1 = 4k^2 - 4k + 1 - 1 = 4k^2 - 4k = 4k(k-1)$. It is clearly a multiple of 4, but since one of the other two factors must be even, it is a multiple of 8.

Problem 1.6

(a) Let us check that the identity holds for $n = 1$. The left-hand side is clearly $a-b$. As for the right-hand side,

$$\sum_{k=1}^{1} a^{1-k}b^{k-1} = a^0 b^0 = 1,$$

so the right-hand side is also $a-b$.
Now let us assume that for a particular n the formula holds. Then

$$a^n = b^n + (a-b)\sum_{k=1}^{n} a^{n-k}b^{k-1}.$$

Multiplying this equation by a we get

$$a^{n+1} = ab^n + (a-b)\sum_{k=1}^{n} a\cdot a^{n-k}b^{k-1} = ab^n + (a-b)\sum_{k=1}^{n} a^{n+1-k}b^{k-1}.$$

Let us substract b^{n+1} from both side of the equation:

$$a^{n+1} - b^{n+1} = ab^n - b^{n+1} + (a-b)\sum_{k=1}^{n} a^{n+1-k}b^{k-1}.$$

In the first two terms of the right-hand side b^n is a common factor, so we can write

$$a^{n+1} - b^{n+1} = b^n(a-b) + (a-b)\sum_{k=1}^{n} a^{n+1-k}b^{k-1},$$

and now $(a-b)$ is a common factor of both term in the right-hand side, so

$$a^{n+1} - b^{n+1} = (a-b)\left[b^n + \sum_{k=1}^{n} a^{n+1-k}b^{k-1}\right] = (a-b)\sum_{k=1}^{n+1} a^{n+1-k}b^{k-1},$$

and we are done, because we have just proven that the formula is also valid for the next natural number $n+1$.

(b) For $n = 1$ we get $n^5 - n = 1 - 1 = 0$, which is trivially a multiple of 5. Now, assuming $n^5 - n$ is a multiple of 5, we can expand

$$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + \not{1} - n - \not{1} = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n).$$

The first term of this sum is a multiple of 5 by assumption, and the second one is obviously a multiple of 5 because of the factor 5 in front of it. Therefore $(n+1)^5 - (n+1)$ will also be a multiple of 5.

(c) For $n = 1$ we have $1 + x \geqslant 1 + x$, which is obviously true. Let us assume that for a certain n it holds $(1+x)^n \geqslant 1 + nx$. Since $x \geqslant -1$, we know that $1 + x \geqslant 0$, so if we multiply the inequality by $(1+x)$ we obtain

$$(1+x)^{n+1} \geqslant (1+x)(1+nx) = 1 + x + nx + nx^2 = 1 + (n+1)x + nx^2.$$

Now, $nx^2 \geqslant 0$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, so $1 + (n+1)x + nx^2 \geqslant 1 + (n+1)x$. Therefore

$$(1+x)^{n+1} \geqslant 1 + (n+1)x.$$

Problem 1.7 In all cases, the first case we must check is $n = 2$, since the inequalities are valid for $n > 1$.

(a) For $n=2$ the left-hand side is $2!=2$ and the right-hand side is $(3/2)^2=9/4=2.25$, so the inequality is true. Now let us prove that

$$n! < \left(\frac{n+1}{2}\right)^n \qquad \Rightarrow \qquad (n+1)! < \left(\frac{n+2}{2}\right)^{n+1}.$$

To that purpose, we multiply the inequality by $n+1$ and get

$$(n+1)! < \frac{(n+1)^{n+1}}{2^n} = 2\left(\frac{n+1}{2}\right)^{n+1}.$$

Then, using the hint,

$$(n+1)! < 2\left(\frac{n+1}{2}\right)^{n+1} < \left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{n+1}{2}\right)^{n+1}.$$

But

$$\left(1+\frac{1}{n+1}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^{n+1},$$

so

$$(n+1)! < 2\left(\frac{n+1}{2}\right)^{n+1} < \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n+1}{2}\right)^{n+1} = \left(\frac{n+2}{2}\right)^{n+1},$$

which proves the inequality we were looking for.

(b) Let us first prove that $(2n+2)! > (n+2)^n(n+2)!$, or equivalently that

$$\frac{(2n+2)!}{(n+2)!} > (n+2)^n.$$

To do that we use the definition of factorial and cancel all common factors in the fraction:

$$\begin{aligned}\frac{(2n+2)!}{(n+2)!} &= \frac{(2n+2)(2n+1)2n\cdots(n+3)\cancel{(n+2)(n+1)n\cdots 3\cdot 2\cdot 1}}{\cancel{(n+2)(n+1)n\cdots 3\cdot 2\cdot 1}} \\ &= (2n+2)(2n+1)\cdots(n+3).\end{aligned}$$

If we now replace all factors by $n+2$, which is smaller than any of them, we get the lower bound

$$\frac{(2n+2)!}{(n+2)!} > \underbrace{(n+2)(n+2)\cdots(n+2)}_{n \text{ times}} = (n+2)^n,$$

which is the equality we wanted to prove.

Let us now prove $2!\cdot 4!\cdots(2n)! > \big[(n+1)!\big]^n$ using induction. Take $n=2$ —the first value for which the inequality is supposed to work. The left-hand side is $2!\cdot 4! = 2\cdot 24 = 48$, while the right-hand side is $(3!)^2 = 6^2 = 36$, so the inequality holds for $n=2$.

Assume now that $2!\cdot 4!\cdots(2n)! > \big[(n+1)!\big]^n$ holds, multiply both sides by $(2n+2)!$ and use the inequality just proven:

$$\begin{aligned}2!\cdot 4!\cdots(2n)!(2n+2)! &> \big[(n+1)!\big]^n(2n+2)! > \big[(n+1)!\big]^n(n+2)^n(n+2)! \\ &= \big[(n+1)!(n+2)\big]^n(n+2)! = \big[(n+2)!\big]^n(n+2)! = \big[(n+2)!\big]^{n+1},\end{aligned}$$

and so the inequality also holds for $n+1$.

(c) Take $n = 2$. The left-hand side is $1 + 1/\sqrt{2} \approx 1.7$, whereas the right-hand side is $\sqrt{2} \approx 1.4$. Hence the inequality holds for the first value of n. Assume now that

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

and add $1/\sqrt{n+1}$ to both sides of the inequality. We end up with

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} = \frac{\sqrt{n(n+1)}+1}{\sqrt{n+1}}.$$

Now, $n(n+1) > n^2$, therefore

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} > \frac{\sqrt{n^2}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

and the inequality is proven.

Notice that in this case there is a simpler way to obtain the inequality *without* using induction. All we need to do to find a lower bound to the sum in the left-hand side is to replace every term by the smallest one, namely $1/\sqrt{n}$. If there are two or more terms in the sum (i.e. if $n > 1$) then we obtain a strict lower bound with this operation. So

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \underbrace{\frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ times}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

Problem 1.8

(a) $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are clearly irrational numbers (their decimal expressions have the same decimal part than $\sqrt{2}$), but if we add them up we obtain 2, which is a rational number.

(b) If we multiply $\sqrt{2}$ by itself we obtain 2.

(c) Consider the number $a = \sqrt{2}^{\sqrt{2}}$, and let us compute

$$a^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2.$$

Therefore, either $a \in \mathbb{Q}$ —in which case $x = y = \sqrt{2}$ is the example we are seeking— or $a \notin \mathbb{Q}$ —in which case $x = a$ and $y = \sqrt{2}$ is that example.

Problem 1.9

(a) Suppose that $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$. If we square both sides,

$$\left(\sqrt{2}+\sqrt{3}\right)^2 = r^2 \quad \Leftrightarrow \quad 2+3+2\sqrt{2}\sqrt{3} = r^2 \quad \Leftrightarrow \quad \sqrt{6} = \frac{r^2-5}{2}.$$

The right-hand side in the last expression is a rational number, so $\sqrt{6}$ must be a rational number. Suppose there is an irreducible fraction p/q such that $\sqrt{6} = p/q$. Squaring, $6q^2 = p^2$, so p must be even, i.e., $p = 2k$. Substituting $6q^2 = 4k^2$, which simplifies to $3q^2 = 2k^2$. Thus q must be even too, but that is not possible because q and p do not have common factors. Hence $\sqrt{6} \notin \mathbb{Q}$ and therefore $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

(b) Suppose $\sqrt{n} = k\sqrt{r} = p/q$, an irreducible fraction. Squaring this equation $rk^2q^2 = p^2$. But this is impossible because r does not contain any square factor whereas all factors in the right-hand side are squared. Hence $\sqrt{n} \notin \mathbb{Q}$.

(c) Suppose $\sqrt{n-1}+\sqrt{n+1}=r\in\mathbb{Q}$. Squaring

$$n-1+n+1+2\sqrt{n-1}\sqrt{n+1}=r^2 \qquad\Leftrightarrow\qquad \sqrt{n^2-1}=\frac{r^2-2n}{2}.$$

The right-hand side is a rational number, but the left-hand side cannot be because n^2-1 is not a perfect square (two consecutive numbers cannot be both perfect squares, and n^2 is). Thus $\sqrt{n-1}+\sqrt{n+1}\notin\mathbb{Q}$.

Problem 1.10 All we need to do is to expand the binomials in the left-hand side of the equation:

$$\left(\frac{x+|x|}{2}\right)^2+\left(\frac{x-|x|}{2}\right)^2=\frac{1}{4}\left[(x+|x|)^2+(x-|x|)^2\right]=\frac{1}{4}[x^2+|x|^2+\cancel{2x|x|}+x^2+|x|^2-\cancel{2x|x|}]$$
$$=\frac{1}{\cancel{4}}\cancel{4}x^2=x^2.$$

Problem 1.11

(i) Since $|z|\leqslant a$ is the same as $-a\leqslant z\leqslant a$,

$$|x-3|\leqslant 8 \quad\Leftrightarrow\quad -8\leqslant x-3\leqslant 8 \quad\Leftrightarrow\quad 3-8\leqslant x\leqslant 3+8 \quad\Leftrightarrow\quad -5\leqslant x\leqslant 11.$$

Hence $A=[-5,11]$.

(ii) On the one hand

$$|x-2|<\frac{1}{2} \quad\Leftrightarrow\quad -\frac{1}{2}<x-2<\frac{1}{2} \quad\Leftrightarrow\quad \frac{3}{2}<x<\frac{5}{2}.$$

On the other hand $0<|x-2|$ holds if, and only if, $x\neq 2$. Therefore $B=\left(\frac{3}{2},2\right)\cup\left(2,\frac{5}{2}\right)$.

(iii) We can factor out $x^2-5x+6=(x-2)(x-3)$, therefore $C=\{x\in\mathbb{R}:\,(x-2)(x-3)\geqslant 0\}$. So C contains those x for which both factors are either nonnegative or nonpositive, i.e., $x\geqslant 3$ and $x\leqslant 2$. Hence $C=(-\infty,2]\cup[3,\infty)$.

(iv) D contains those x for which an odd numbers of the three factors in the inequality are negative, i.e. $x<-3$ and $0<x<5$. Thus $D=(-\infty,-3)\cup(0,5)$.

(v) Factoring out $x^2+8x+7=(x+1)(x+7)$ we can rewrite

$$E=\left\{x\in\mathbb{R}:\,\frac{2(x+4)}{(x+1)(x+7)}>0\right\}.$$

Thus either all factors in the fraction must be positive or one positive and the other two negatives. This holds for $x>-1$ and $-7<x<-4$, hence $E=(-7,-4)\cup(-1,\infty)$.

(vi) Since

$$\frac{4}{x}<x \quad\Leftrightarrow\quad 0<x-\frac{4}{x} \quad\Leftrightarrow\quad 0<\frac{x^2-4}{x} \quad\Leftrightarrow\quad 0<\frac{(x-2)(x+2)}{x},$$

F will contain those x for which either all three factors in the fraction are positive or one is positive and two negatives, i.e., $2<x$ and $-2<x<0$. Hence $F=(-2,0)\cup(2,\infty)$.

(vii) The inequality $4x<2x+1$ is equivalent to $2x<1$, i.e., $x<\frac{1}{2}$. The inequality $2x+1\leqslant 3x+2$ is equivalent to $0\leqslant x+1$, i.e., $-1\leqslant x$. Hence $G=[-1,\frac{1}{2})$.

(viii) $|x^2-2x|<1$ means $-1<x^2-2x<1$. The inequality $-1<x^2-2x$ means $0<x^2-2x+1=(x-1)^2$, which only holds for $x\neq 1$. On the other hand, the inequality $x^2-2x<1$ means $x^2-2x-1<0$, which holds for all x within the two roots of $x^2-2x-1=0$. These two roots are $1+\sqrt{2}>0$ and $1-\sqrt{2}<0$. Therefore $H=(1-\sqrt{2},1)\cup(1,1+\sqrt{2})$.

(ix) The equation

$$|x-1||x+2| = 10 \quad \Leftrightarrow \quad |(x-1)(x+2)| = 10 \quad \Leftrightarrow \quad |x^2+x-2| = 10$$

is actually two in one, namely

$$x^2+x-2 = 10, \qquad x^2+x-2 = -10.$$

The solutions of the first one are the solutions of $x^2+x-12=0$, i.e., $x=-4$ and $x=3$. On the other hand, the second equation becomes $x^2+x+8=0$, which has no real solutions. Thus $I = \{-4\} \cup \{3\}$.

(x) The inequality $|x-1|+|x+2| > 1$ has to be discussed in three regions: (a) $x \geqslant 1$, (b) $-2 \leqslant x < 1$, and (c) $x \leqslant -2$.
(a) $x \geqslant 1$. The inequality becomes $x-1+x+2 > 1$ because the numbers within the absolute values are both nonnegatives. This is equivalent to $2x+1 > 1$, i.e., $x > 0$. Since we are assuming that $x \geqslant 1$, all numbers in this region satisfy the inequality.
(b) $-2 \leqslant x < 1$. The inequality becomes $1-x+x+2 > 1$ since $x-1 < 0$ but $x+2 \geqslant 0$. This inequality turns out to be $3 > 1$, which is obviously true, so all numbers in this region satisfy the inequality.
(c) $x < -2$. The inequality becomes $1-x-x-2 > 1$ since both $x-1 < 0$ and $x+2 < 0$. This inequality becomes $-2x-1 > 1$, i.e., $2+2x < 0$ or $x < -1$. But we are in a region where $x < -2$, so all numbers in this region satisfy $x < -1$.
Consequently $J = \mathbb{R}$.

Problem 1.12

(i) $x(0) = a$, $x(1) = b$, $x(1/2) = (a+b)/2$.

(ii) $B = (a,b)$.

(iii) $C = (-\infty, a)$.

(iv) $D = (b, \infty)$.

Problem 1.13

(i) $\sup A = 3 \neq \max A$; $\inf A = -1 = \min A$.

(ii) $\sup B = 3 = \max B$; $\inf B = -1 = \min B$.

(iii) $\sup C = 3 = \max C$; $\inf C = 2 \neq \min C$.

(iv) Writing $(n^2+1)/n$ as $n+1/n$ is clear that $\sup D = \infty$; $\inf D = 2 = \min D$.

(v) The two roots of the parabola are $x=3$ and $x=1/3$. Since the coefficient of x^2 is positive, the parabola is negative between the two roots. Hence $E = (1/3, 3)$ and $\sup E = 3 \neq \max E$; $\inf E = 1/3 \neq \min E$.

(vi) F contains those numbers for which an odd number of factors are negative. Thus $F = (a,b) \cup (c,d)$ and $\sup F = d \neq \max F$; $\inf F = a \neq \min F$.

(vii) $\sup G = \frac{1}{2} + \frac{1}{5} = \frac{7}{10} = \max G$; $\inf G = 0 \neq \min G$.

(viii) We can express $H = H_+ \cup H_-$, where $H_+ = \{1+1/m : m \in \mathbb{N}\}$ and $H_- = \{-1+1/m : m \in \mathbb{N}\}$. Since all numbers in H_- are smaller than all numbers in H_+, $\sup H = \sup H_+ = 2 = \max H$, whereas $\inf H = \inf H_- = -1 \neq \min H$.