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Calculus I

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Unit 2. Real Functions

Solutions

D.2 Real Functions

Problem 2.1

(i) We can factor out the denominator as $x^2-5x+6=(x-2)(x-3)$; therefore, the domain is $\mathbb{R}-\{2,3\}$.

(ii) There are two conditions for $f(x)$ to exist: $1-x^2\geqslant 0$ and $x^2-1\geqslant 0$. Together they imply $1-x^2=0$. Therefore the domain is just the set $\{-1,1\}$.

(iii) There are two conditions to be met for x to be in the domain: first, $1-x^2\geqslant 0$; second, $x\neq\sqrt{1-x^2}$. The first condition implies $x^2\leqslant 1$, or equivalently, $-1\leqslant x\leqslant 1$. The second condition is not fulfilled if $x=\sqrt{1-x^2}$. Squaring this equation we obtain $x^2=1-x^2$, which is equivalent to $x^2=1/2$. The two solutions of this equation are $x=\pm 1/\sqrt{2}$, but of them two, only the positive one is a solution of the original equation $x=\sqrt{1-x^2}$. Thus the domain is $[-1,1/\sqrt{2})\cup(1/\sqrt{2},1]$.

(iv) The two coditions to be met for x to be in the domain are $4-x^2\geqslant 0$ and $1-\sqrt{4-x^2}\geqslant 0$. The first one reads $x^2\leqslant 4$, i.e., $-2\leqslant x\leqslant 2$. The second one implies $\sqrt{4-x^2}\leqslant 1$. Both sides of this inequality are positive, so we can square it to obtain $4-x^2\leqslant 1$, i.e., $x^2\geqslant 3$. This holds either if $x\geqslant\sqrt{3}$ or $x\leqslant-\sqrt{3}$. Therefore, the domain is $[-2,-\sqrt{3}]\cup[\sqrt{3},2]$.

(v) The denominator vanishes if $\log x=1$, i.e., if $x=e$. Since the logarithm requires $x>0$, the domain is $(0,e)\cup(e,\infty)$.

(vi) The condition to be met now is $x-x^2>0$. We can factor $x-x^2=x(1-x)$, so the roots of the parabola are $x=0$ and $x=1$. Since the coefficient of x^2 is negative, the parabola is positive provided $0<x<1$. The domain is then $(0,1)$.

(vii) Three conditions need to be met: first, $x>0$ because x is the argument of a logarithm; second, $\log x\neq 0$ because it is the denominator; and third, $5-x\geqslant 0$ because it is the argument of a square root. The second condition implies $x\neq 1$, whereas the third one implies $x\leqslant 5$. Thus the domain is $(0,1)\cup(1,5]$.

(viii) The first requirement is $x>0$ for the logarithm to be meaningful. The second one is $-1\leqslant\log x\leqslant 1$ because the domain of the arcsin is the interval $[-1,1]$. Since $\log x$ is monotonically increasing, this inequality is equivalent to $e^{-1}\leqslant x\leqslant e$, so the domain is the interval $[e^{-1},e]$.

Problem 2.2

(a) We know that $f(-x)=-f(x)$ and $g(-x)=-g(x)$. Then

$$(f+g)(-x)=f(-x)+g(-x)=-f(x)-g(-x)=-(f+g)(x),$$

so $f+g$ is *odd*. Now,

$$(fg)(-x)=f(-x)g(-x)=[-f(x)][-g(x)]=f(x)g(x)=(fg)(x),$$

so fg is *even*. Finally,

$$(f\circ g)(-x)=f\big(g(-x)\big)=f\big(-g(x)\big)=-f\big(g(x)\big)=-(f\circ g)(x).$$

Thus $f\circ g$ is *odd*.

(b) Now $f(-x)=f(x)$ and $g(-x)=-g(x)$. Then

$$(f+g)(-x)=f(-x)+g(-x)=f(x)-g(-x),$$

so $f+g$ is neither even nor odd. As for the product,

$$(fg)(-x)=f(-x)g(-x)=f(x)[-g(x)]=-f(x)g(x)=-(fg)(x),$$

so fg is *odd*. Finally,

$$(f\circ g)(-x)=f\big(g(-x)\big)=f\big(-g(x)\big)=f\big(g(x)\big)=(f\circ g)(x).$$

Thus $f\circ g$ is *even*.

Problem 2.3 (i)

$$f(-x)=\frac{-x}{(-x)^2+1}=-f(x).$$

The function is *odd*.

(ii)

$$f(-x)=\frac{(-x)^2-(-x)}{(-x)^2+1}=\frac{x^2+x}{x^2+1}\neq\pm f(x),$$

so the function is neither.

(iii)

$$f(-x)=\frac{\sin(-x)}{-x}=\frac{-\sin x}{-x}=\frac{\sin x}{x}=f(x).$$

The function is *even*.

(iv)

$$f(-x)=\cos\big((-x)^3\big)\sin\big((-x)^2\big)e^{-(-x)^4}=\cos(-x^3)\sin(x^2)e^{-x^4}=\cos(x^3)\sin(x^2)e^{-x^4}=f(x).$$

The function is *even*.

(v)

$$f(-x)=\frac{1}{\sqrt{(-x)^2+1}-(-x)}=\frac{1}{\sqrt{x^2+1}+x},$$

so the function is neither.

(vi) This function is the logarithm of the function in the previous item, so it seems that it has no defined parity because

$$f(-x)=\log\left(\sqrt{x^2+1}+x\right).$$

However,

$$\sqrt{x^2+1}+x=\frac{\left(\sqrt{x^2+1}+x\right)\left(\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}-x}=\frac{x^2+1-x^2}{\sqrt{x^2+1}-x}=\frac{1}{\sqrt{x^2+1}-x},$$

so

$$f(-x)=\log\left(\sqrt{x^2+1}+x\right)=\log\left(\frac{1}{\sqrt{x^2+1}-x}\right)=-\log\left(\sqrt{x^2+1}-x\right)=-f(x).$$

The function is *odd*.

Problem 2.4 The equation defining the inverse function is $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. If the function is to be its own inverse it must satisfy the equation $(f \circ f)(x) = x$. In other words,

$$(f \circ f)(x) = \frac{af(x)+b}{cf(x)+d} = \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} = \frac{a(ax+b)+b(cx+d)}{c(ax+b)+d(cx+d)} = \frac{(a^2+bc)x+(a+d)b}{(a+d)cx+(bc+d^2)} = x.$$

For the last two expressions to be the same we must have

$$(a+d)c = (a+d)b = 0,$$
$$a^2 = d^2.$$

There are two ways in which the top equations can be fulfilled. The first one is $c = b = 0$. Since the second equation implies $a = \pm d$, the two resulting functions are $f_1(x) = x$ and $f_2(x) = -x$. The second possibility is that $a+d = 0$, or $d = -a$. Then all three equations hold. This corresponds to the function

$$f(x) = \frac{ax+b}{cx-a},$$

whose only constraint is that c and a cannot be both zero.

Problem 2.5 The statement of the problem is that $f(x)$, understood as a mapping $f : \mathbb{R} - \{-1/2\} \mapsto \mathbb{R} - \{1/2\}$, is bijective. A simple way to see that the domain of f is $\mathbb{R} - \{-1/2\}$, that it can be inverted in its domain, and that the domain of f^{-1} is $\mathbb{R} - \{1/2\}$.

That the domain is $\mathbb{R} - \{-1/2\}$ is obvious because $x = -1/2$ is the only zero of the denominator. That f can be inverted is a matter of solving x as a function of y in the equation

$$y = \frac{x+3}{1+2x} \quad \Rightarrow \quad y(1+2x) = x+3 \quad \Rightarrow \quad y-3 = x(1-2y) \quad \Rightarrow \quad x = \frac{y-3}{1-2y}.$$

The inverse function is then

$$f^{-1}(x) = \frac{x-3}{1-2x}$$

and its domain is clearly $\mathbb{R} - \{1/2\}$.

Problem 2.6

(a) An easy way to check for injectivity is to determine whether the equation $y = f(x)$ has a unique solution for those y for which it can be solved.

(i) For every $y \in \mathbb{R}$,

$$y = 7x-4 \quad \Rightarrow \quad x = \frac{y+4}{7}.$$

So there is a unique solution no matter y, which means that the function is injective.

(ii) Only if $-1 \leqslant y \leqslant 1$ the equation

$$y = \sin(7x-4)$$

can have a solution. On the other hand, two points x_1 and x_2 such that $7x_2 - 4 = 7x_1 - 4 + 2n\pi$, with $n \in \mathbb{Z}$, are both solutions of the same y. Clearly $x_2 = x_1 + 2n\pi/7$. Therefore there are infinitely many solutions for each $-1 \leqslant y \leqslant 1$, which means that the function is not injective.

(iii) For any $y \in \mathbb{R}$,

$$y = (x+1)^3 + 2 \quad \Rightarrow \quad x = (y-2)^{1/3} - 1,$$

so the solution is unique and the function is injective.

(iv) Take y so that

$$y = \frac{x+2}{x+1}.$$

Then

$$y(x+1) = x+2 \quad \Rightarrow \quad y-2 = x(1-y).$$

Thus, provided $y \neq 1$, we obtain

$$x = \frac{y-2}{1-y}$$

and the solution is unique. The function is injective.

(v) Take y and solve for $y = x^2 - 3x + 2$, or $x^2 - 3x + 2 - y = 0$. Then

$$x = \frac{3 \pm \sqrt{9 + 4(y-2)}}{2} = \frac{3 \pm \sqrt{4y+1}}{2}.$$

The equation has a solution only if $y \geqslant -1/4$. But for all $y > -1/4$ there are two different solutions. Therefore the function is not injective.

(vi) Consider the equation

$$y = \frac{x}{x^2+1}.$$

If $y = 0$ the only solution is $x = 0$. If $y \neq 0$ it can be transformed into

$$y(x^2+1) = x \quad \Rightarrow \quad yx^2 - x + y = 0.$$

The solutions of this quadratic equation are

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}.$$

There is solution only if $y^2 \leqslant 1/4$, i.e., $-1/2 \leqslant y \leqslant 1/2$, but for every $-1/2 < y < 1/2$ there are two different solutions for the same y, hence the function is not injective.

(vii) For every $y > 0$,

$$y = e^{-x} \quad \Rightarrow \quad \log y = -x \quad \Rightarrow \quad x = -\log y.$$

The solution is unique and the function is injective.

(viii) For every $y \in \mathbb{R}$,

$$y = \log(x+1) \quad \Rightarrow \quad e^y = x+1 \quad \Rightarrow \quad x = e^y - 1.$$

The solution is unique and the function is injective.

(b) The solutions of the equation $y = x^2 - 3x + 2$ are (see previous item)

$$x = \frac{3 \pm \sqrt{4y+1}}{2}.$$

Clearly one solution is larger than $3/2$ and the other is smaller than $3/2$. Therefore, if we limit the domain to those x larger than $3/2$ only one solution survives and the function becomes injective.

(c) Take two values $1 < x_1 < x_2$ and calculate the difference

$$f(x_1) - f(x_2) = \frac{x_1}{x_1^2+1} - \frac{x_2}{x_2^2+1} = \frac{x_1(x_2^2+1) - x_2(x_1^2+1)}{(x_1^2+1)(x_2^2+1)} = \frac{(x_2-x_1)(x_1x_2-1)}{(x_1^2+1)(x_2^2+1)} > 0.$$

So $f(x)$ is monotonically decreasing for $x > 1$ and therefore injective.
Now, since the solution of $y = f(x)$ is (see previous item)

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y},$$

for $y = \sqrt{2}/3$,

$$x = \frac{1 \pm \sqrt{1-8/9}}{2\sqrt{2}/3} = \frac{3 \pm 1}{2\sqrt{2}} = \begin{cases} \sqrt{2}, \\ \dfrac{1}{\sqrt{2}}. \end{cases}$$

Since the domain of the function is $(1, \infty)$, only the top solution is in the domain; thus $f^{-1}(\sqrt{2}/3) = \sqrt{2}$.

(d)

(i) There is a unique solution for every $y \in \mathbb{R}$, therefore the function is surjective, hence bijective.

(ii) Not surjective because the range is $[-1, 1]$.

(iii) Surjective and bijective.

(iv) Not surjective because $y = 1$ is not in the range of the function.

(v) Not surjective because the range is $[-1/4, \infty)$.

(vi) Not surjective because the range is $[-1/2, 1/2]$.

(vii) Not surjective because the range is $(0, \infty)$.

(viii) Surjective and bijective.

Problem 2.7

(i) Let us denote $\theta_1 = \arctan\frac{1}{2}$, $\theta_2 = \arctan\frac{1}{3}$, and $\theta = \theta_1 + \theta_2$. Firts of all, $0 < \theta_{1,2} < \pi/4$, so $0 < \theta < \pi/2$. Now,

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \frac{5/6}{5/6} = 1.$$

Therefore $\theta = \pi/4$.

(ii) Now $\theta_1 = \arctan 2$ and $\theta_2 = \arctan 3$, and $\pi/4 < \tan\theta_{1,2} < \pi/2$, so $\pi/2 < \tan\theta < \pi$. Then

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{2+3}{1-6} = \frac{5}{-5} = -1.$$

Therefore $\theta = 3\pi/4$.

(iii) Denote $\theta_1 = \arctan\frac{1}{2}$, $\theta_2 = \arctan\frac{1}{5}$, $\theta_3 = \arctan\frac{1}{8}$, and $\theta = \theta_1 + \theta_2 + \theta_3$. Since $0 < \tan\theta_i < \pi/4$ then $0 < \tan\theta < 3\pi/4$. Accordingly $\tan\theta > 0$ if $0 < \theta < \pi/2$ and $\tan\theta < 0$ if $\pi/2 < \theta < 3\pi/4$.
First of all we need to work out a formula for $\tan(\theta_1 + \theta_2 + \theta_3)$. For the sake of simplicity we will denote $\tau_i = \tan\theta_i$ and $\tau_{ij} = \tan(\theta_i + \theta_j)$. Thus,

$$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{\tau_{12} + \tau_3}{1 - \tau_{12}\tau_3} = \frac{\frac{\tau_1+\tau_2}{1-\tau_1\tau_2} + \tau_3}{1 - \frac{\tau_1+\tau_2}{1-\tau_1\tau_2}\tau_3} = \frac{\tau_1 + \tau_2 + \tau_3 - \tau_1\tau_2\tau_3}{1 - \tau_1\tau_2 - \tau_2\tau_3 - \tau_3\tau_1}.$$

The formula is

$$\tan\theta = \frac{\tan\theta_1 + \tan\theta_2 + \tan\theta_3 - \tan\theta_1\tan\theta_2\tan\theta_3}{1 - \tan\theta_1\tan\theta_2 - \tan\theta_2\tan\theta_3 - \tan\theta_3\tan\theta_1}.$$

substituting,

$$\tan\theta = \frac{\frac{1}{2} + \frac{1}{5} + \frac{1}{8} - \frac{1}{80}}{1 - \frac{1}{10} - \frac{1}{40} - \frac{1}{16}} = \frac{40 + 16 + 10 - 1}{80 - 8 - 2 - 5} = \frac{65}{65} = 1.$$

Thus $\theta = \pi/4$.

Problem 2.8

(i) Denote $\theta = \arccos x$, so that $\cos\theta = x$. Then

$$f(x) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - x^2}.$$

(ii) Denote $\theta = \arcsin x$, so that $\sin\theta = x$. Then,

$$f(x) = \sin(2\theta) = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - \sin^2\theta} = 2x\sqrt{1 - x^2}.$$

(iii) Denote $\theta = \arccos x$, so that $\cos\theta = x$. Then

$$f(x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1 - \cos^2\theta}}{\cos\theta} = \frac{\sqrt{1 - x^2}}{x}.$$

(iv) Denote $\theta = \arctan x$, so that $\tan\theta = x$. Now,

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} - 1,$$

so

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}}.$$

And since $\sin\theta = \cos\theta\tan\theta$,

$$\sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}.$$

Then

$$f(x) = \sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}.$$

(v) Denote $\theta = \arctan x$, so that $\tan\theta = x$. Then

$$f(x) = \cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - x^2}{1 + x^2}.$$

(vi) Since $4\log x = \log(x^4)$, then $f(x) = e^{\log(x^4)} = x^4$.

Problem 2.9 If we take logarithms in the first of these equations we obtain

$$y\log x = x\log y.$$

The second equation is $y = 3x$, so substituting for y in the previous equation we end up with

$$3x\log x = x\log(3x) = x\log 3 + x\log x.$$

Since $x > 0$ (i.e., $x \neq 0$) we can cancel one factor x from the equation,

$$3\log x = \log 3 + \log x \quad \Rightarrow \quad 2\log x = \log 3 \quad \Rightarrow \quad \log x = \frac{1}{2}\log 3 \quad \Rightarrow \quad \log x = \log\sqrt{3},$$

therefore $x = \sqrt{3}$.

Problem 2.10 Use GeoGebra to help you with this exercise.

Problem 2.11 Here are some hints to help you plot these functions:

(i) Start off with the plot of $g(x) = x^2$; function $f(x) = g(x+2) - 1$, so shift the plot two units to the left and one unit down.

(ii) Start off with the plot of $g(x) = \sqrt{x}$ and then tranform it into that of $h(x) = \sqrt{-x}$ by reflecting it on the Y axis. Then $f(x) = h(x-4)$, so shift this plot four units to the right.

(iii) Start off from the plots of $g_1(x) = x^2$ and $g_2(x) = 1/x$. Near $x = 0$ g_1 is negligible with respect to g_2 —which diverges to $\pm\infty$ at $x = 0$. Far from $x = 0$ it is g_2 that is negligible with respect to g_1, which grows indefinitely. So $f(x)$ is close to $g_2(x)$ as x 'moves' toward 0, and close to $g_1(x)$ as x goes far awat from $x = 0$. Sketch the plot of $f(x)$ using this information.

(iv) Start off with the plot of $g(x) = x^2$ and shift it up one unit to get that of $h(x) = x^2 + 1$. Then $f(x) = 1/h(x)$. Since $h(x) > 1$ for all $x \neq 0$ and $h(0) = 1$, then $f(x) < 1$ for all $x \neq 0$ and $f(0) = 1$. Besides, $h(x)$ grows indefinitely as x goes away from the origin, so $f(x)$ has to approach 0.

(v) $g(x) = x - x^2 = x(1-x)$, so $g(x) > 0$ if $0 < x < 1$ and $g(x) < 0$ if $x < 0$ or $x > 1$. Therefore

$$f(x) = \begin{cases} x^2, & \text{if } 0 \leqslant x \leqslant 1, \\ x, & \text{otherwise.} \end{cases}$$

(vi) e^x is monotonically increasing and croses 1 at $x = 0$. Therefore

$$f(x) = \begin{cases} e^x - 1, & \text{if } x \geqslant 0, \\ 1 - e^x, & \text{if } x < 0. \end{cases}$$

All that needs to be done is to reflect the graph of $e^x - 1$ (equal to that of e^x but shifted down one unit) for $x < 0$ on the X axis.

(vii) If $x \geqslant 0$ then $|x| - x = 0$, but if $x < 0$ then $|x| - x = -2x$. So

$$f(x) = \begin{cases} 0, & \text{if } x \geqslant 0, \\ \sqrt{-2x}, & \text{if } x < 0. \end{cases}$$

(viii) $\lfloor x \rfloor$ is the smallest integer not larger than x. So for instance, $\lfloor 3.14 \rfloor = 3$, $\lfloor 0.5 \rfloor = 0$, but $\lfloor -1.58 \rfloor = -2$. So the function $f(x)$ is going to be piecewise constant, equal to $1/n$ for some integer n, in given intervals. All we need to do is to determine those intervals. Of course, $f(x)$ is only defined if $x \neq 0$ and if $\lfloor \frac{1}{x} \rfloor \neq 0$.

Let n be an integer and let us try to figure out where

$$\left\lfloor \frac{1}{x} \right\rfloor = n.$$

By definition

$$n \leqslant \frac{1}{x} < n+1. \tag{D.1}$$

As we have mentioned above, $f(x)$ will not be defined if $n = 0$. This means all x such that

$$0 \leqslant \frac{1}{x} < 1.$$

The left inequality implies $x > 0$. The right inequality implies $x > 1$. Therefore the domain of f is $(-\infty, 0) \cup (0, 1]$.
Consider first $x \in (0, 1]$. Then, according to (D.1) $n > 0$. From the left inequality $x \leqslant 1/n$, and from the right one $x > 1/(n+1)$. Thus

$$f(x) = \frac{1}{n} \quad \text{for all } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \qquad n \in \mathbb{N}.$$

In other words, $f(x) = 1$ for $x \in (1/2, 1]$, $f(x) = 1/2$ for $x \in (1/3, 1/2]$, $f(x) = 1/3$ for $x \in (1/4, 1/3]$, etc. This covers the plot of $f(x)$ within the interval $(0, 1]$. By the way, the function gets closer and closer to 0 as x approaches 0.
Consider now the interval $(-\infty, 0)$. Then n in (D.1) must be negative. Then the left inequality again implies $x \leqslant 1/n$ and the right one $x > 1/(n+1)$. The result is the same:

$$f(x) = \frac{1}{n} \quad \text{for all } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \qquad n \in -\mathbb{N}.$$

So we have $f(x) = -1$ if $x \in (-\infty, -1]$, $f(x) = -1/2$ if $x \in (-1, -1/2]$, $f(x) = -1/3$ if $x \in (-1/2, -1/3]$, etc. This covers the whole interval $(-\infty, 0)$.

(ix) Function $g(x) = x^2 - 1 < 0$ if $-1 < x < 1$ and $g(x) > 0$ otherwise, so

$$f(x) = \begin{cases} 1 - x^2, & \text{if } -1 < x < 1, \\ x^2 - 1, & \text{otherwise.} \end{cases}$$

All that one has to do is to reflect the portion of the graph of $x^2 - 1$ in the interval $(-1, 1)$ on the X axis.

(x) Plot $g(x) = e^x$. The plot of $g(-x)$ is just the mirror image with respect to the Y axis. And that of $-g(-x)$ is a new reflection with respect to the X axis. Shift the whole plot one unit upward and you will get the plot of $f(x) = -g(-x) + 1 = 1 - e^{-x}$.

(xi) The function is defined only if $|x| \geqslant 1$. Besides, it is an even function, so it will be symmetric with respect to the Y axis. Let us then focus on the positive interval $[1, \infty)$. Notice that $f(x) = \log(x-1) + \log(x+1)$. These are two graphs of $\log x$, the first one shifted one unit to the right and the second one shifted one unit to the left. Since $\log x$ grows very slowly but diverges at $x = 0$, near the point $x = 1$ function $\log(x-1)$ will diverge and $\log(x+1)$ will then be negligible. In oher words, $f(x) \approx \log(x-1)$. On the other hand, when x is large $x \pm 1 \approx x$, so $f(x) \approx 2\log x$. Plot $f(x)$ using this information.

(xii) As x grows far away from the origin (positive or negative) $1/x$ becomes very small, so $\sin(1/x)$ approaches $1/x$, and therefore $f(x)$ approaches 1. On the other hand, $\sin(1/x)$ oscillates wildly as x gets near the origin, but x modulates the amplitude (making it smaller the closer to the origin).

Problem 2.12

(a) Since $\cosh x = (e^x + e^{-x})/2$, $\sinh x = (e^x - e^{-x})/2$,

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x, \qquad \sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh x.$$

(b) First identity:

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) - \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) = 1.$$

Second identity:

$$2\sinh x \cosh x = \frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) = \frac{1}{2}\left(e^{2x} - 1 + 1 - e^{-2x}\right) = \sinh(2x).$$