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Calculus I

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Unit 3. Sequences

Exercises

Problems

Problem 3.1

(a) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence and let $\{y_n\}_{n=1}^{\infty}$ be a divergent sequence. What can be said of the produt sequence $\{x_n y_n\}_{n=1}^{\infty}$.

(b) If a sequence of integer numbers is convergent, what is this sequence like?

(c) Prove that every convergent sequence is bounded.

Problem 3.2 Given the following recurrent sequences, find the general term and compute their limit:

(i) $a_{n+1} = \dfrac{a_n + 1}{2}$, with $a_0 = 0$; (ii) $b_{n+1} = \sqrt{2b_n}$, with $b_0 = 1$.

Problem 3.3 Calculate the following limits:

(i) $\lim_{n\to\infty} \sqrt[n]{a^n + b^n}$, with $a, b > 0$;

(ii) $\lim_{n\to\infty} \left(\dfrac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n$, with $a, b > 0$;

(iii) $\lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right)$;

(iv) $\lim_{n\to\infty} \sqrt{n}\left(\sqrt[4]{n^2+1} - \sqrt{n+1}\right)$;

(v) $\lim_{n\to\infty} \dfrac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$;

(vi) $\lim_{n\to\infty} \left(\dfrac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}$.

Problem 3.4 Calculate the following limits:

(i) $\lim_{n\to\infty} \dfrac{n}{\pi} \sin n\pi$;

(ii) $\lim_{n\to\infty} \dfrac{n\left(e^{1/n} - e^{\sin(1/n)}\right)}{1 - n\sin(1/n)}$;

(iii) $\lim_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}}$;

(iv) $\lim_{n\to\infty} n^{-3/n}$;

(v) $\lim_{n\to\infty} \dfrac{2^n}{n!}$;

(vi) $\lim_{n\to\infty} \dfrac{n^2}{2^n}$;

(vii) $\lim_{n\to\infty} \dfrac{n^{n-1}}{(n-1)^n}$;

(viii) $\lim_{n\to\infty} \dfrac{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \cdots + n\sqrt[n]{n}}{n^2}$.

Problem 3.5 If $a > 0$ and $\lim_{n\to\infty} u_n = 0$, calculate the following limits:

(i) $\lim_{n\to\infty} \left(\cos\dfrac{b}{n} + a\sin\dfrac{b}{n}\right)^n$; (ii) $\lim_{n\to\infty} \sqrt[u_n]{\dfrac{a - bu_n}{a + bu_n}}$.

Problem 3.6 Calculate the following limits:

(i) $\lim_{n\to\infty} \dfrac{\sum_{k=1}^{n} \sin(\pi/k)}{\log n}$; (ii) $\lim_{n\to\infty} \prod_{k=1}^{n} (2k-1)^{1/n^2}$; (iii) $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{k^2}{n^2} \sin\dfrac{1}{k}$.

Problem 3.7 Given that $\lim_{n\to\infty} a_n = a$, calculate

$$\lim_{n\to\infty} \frac{a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}}{\log(n+1)}.$$

Problem 3.8 Calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}}$$

using the sandwich rule.
HINT: Use the largest and smallest terms in the sum to bound the sum from above and from below, respectively.

Problem 3.9 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_n - n) = \ell$.

(a) Prove that $\lim_{n\to\infty} \frac{a_n}{n} = 1$.

(b) Prove that $\lim_{n\to\infty} n\log(a_n/n) = \ell$.

Problem 3.10 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_{n+1}/a_n) = \ell$. Apply Stolz theorem to calculate the limit

$$\lim_{n\to\infty} \sqrt[n^2]{\frac{a_n^n}{a_1 a_2 \cdots a_n}}$$

Problem 3.11 Prove that the following sequences are monotonic, determine whether they are bounded, and find the limit in case they are:

(i) $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots;$

(ii) $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots;$

(iii) $u_{n+1} = 3 + \frac{u_n}{2}$, with $u_0 = 0$;

(iv) $u_{n+1} = 3 + 2u_n$, with $u_0 = 0$;

(v) $u_{n+1} = \frac{u_n^3+6}{7}$, with (a) $u_0 = 1/2$, (b) $u_0 = 3/2$, and (c) $u_0 = 3$.

Problem 3.12 Consider the sequence defined by $a_{n+1} = \sqrt{1+3a_n} - 1$, with $a_0 = 1/2$.

(a) Prove that the sequence has a limit and find it.

(b) Compute $\lim_{n\to\infty} \frac{a_{n+1}-1}{a_n - 1}$.

Problem 3.13 Consider the sequence defined by $b_{n+1} = 1 - b_n/2$, with $b_0 = 0$.

(a) Prove that the sequence is alternating, i.e., $(b_{n+1}-b_n)(b_n - b_{n-1}) < 0$.

(b) Assuming that it has a limit ℓ, find it.

(c) Prove that $|b_{n+1} - \ell| = \frac{1}{2}|b_n - \ell|$.

(d) Prove that the sequence has indeed a limit.

Problem 3.14 Consider the sequence defined by

$$x_{n+1} = \frac{x_n(1+x_n)}{1+2x_n}, \qquad x_1 = 1.$$

(a) Prove that $x_n > 0$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence is monotonically decreasing.

(c) Calculate its limit.