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Calculus I

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Unit 3. Sequences

Solutions

D.3 Sequences

Problem 3.1

(a) If $\lim_{n\to\infty} x_n = 0$ then it is a $0\cdot\infty$ indeterminacy and the result will depend on the sequences involved. For instance, if $x_n = 1/n$ and $y_n = n$, then $x_n y_n = 1$ is a convergent series. Or if $y_n = n^2$ instead, then $x_n y_n = n$ diverges. Or if $x_n = (-1)^n/n$ and $y_n = n$ then $x_n y_n = (-1)^n$ which is neither convergent nor divergent.
However, if $\lim_{n\to\infty} x_n = a \neq 0$, then we are certain that $x_n y_n$ will diverge. To prove that suppose that $a > 0$ (if $a < 0$ the argument is analogous). Take some $0 < \varepsilon < a$ and chose $C > 0$ arbitrarily large. For sufficently large n we will have at the same time $a-\varepsilon < x_n < a+\varepsilon$ and $C < y_n$. Thus $0 < (a-\varepsilon)C < x_n y_n$, and it is clear that $(a-\varepsilon)C$ can be made arbitrarily large by suitably choosing C.

(b) There will be an index $N \in \mathbb{N}$ such that the sequence is constant for all $n > N$. To prove it just choose a very small $\varepsilon > 0$. If the limit of the sequence is ℓ, then $\ell-\varepsilon < x_n < \ell+\varepsilon$ for all $n > N$. But $x_n \in \mathbb{Z}$, so the only way that thisn inequality can hold for very small ε is that $\ell \in \mathbb{Z}$ and that $x_n = \ell$ for all $n > N$.

(c) By definition, if we take some $\varepsilon > 0$ there will be an $N \in \mathbb{N}$ such that $\ell-\varepsilon < x_n < \ell+\varepsilon$ for all $n > N$ (ℓ is the limit). Then the sequence is bounded for $n > N$. There remain $\{x_1, x_2, \cdots, x_N\}$ outside that interval. But there is a finite number of these numbers, so one of them is the largest (say x_j) and another one is the smallest (say x_k). Define $a = \min\{x_k, \ell-\varepsilon\}$ and $b = \max\{x_j, \ell+\varepsilon\}$. Then for all $n \in \mathbb{N}$ we have $a \leqslant x_n \leqslant b$, which proves that the sequence is bounded.

Problem 3.2

(i) Let us compute a few terms of the sequence:

$$\left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots\right\}.$$

The pattern is clearly

$$a_n = \frac{2^n - 1}{2^n} = 1 - 2^{-n}.$$

Clearly this holds for $a_0 = 1 - 1 = 0$, and if we introduce this formula in the recurrence we obtain

$$a_{n+1} = \frac{a_n + 1}{2} = \frac{1 - 2^{-n} + 1}{2} = \frac{2 - 2^{-n}}{2} = 1 - 2^{-(n+1)}.$$

This proves the guessed formula by induction.
The limit of the sequence a_n is clearly 1.

(ii) Define $a_n = \log_c b_n$, where c is the base of the logarithm —to be specified. If we take $\log_c$ in both sides of the equation we obtain

$$\log_c b_{n+1} = \log_c\left(\sqrt{2b_n}\right) = \frac{1}{2}(\log_c 2 + \log_c b_n) \quad \Leftrightarrow \quad a_{n+1} = \frac{\log_c 2 + a_n}{2},$$

and obviously $a_0 = \log_c b_0 = \log_c 1 = 0$. Now, if we take $c = 2$ (binary logarithms) then $\log_2 2 = 1$ and recurrence we obtain is

$$a_{n+1} = \frac{a_n + 1}{2}, \qquad a_0 = 0,$$

exactly the same as in item (a). The solution will therefore be the same, and

$$b_n = 2^{a_n} = 2^{1-2^{-n}}.$$

The limit of b_n is clearly 2.

Problem 3.3

(i) Applying Corollary 3.4.4 to Stolz's theorem, instead of calculating the limit of $\sqrt[n]{a_n}$ we will calculate the limit of a_n/a_{n-1}. Thus we want to obtain the limit

$$\lim_{n\to\infty} \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \ell.$$

Suppose $a = b$. Then

$$\ell = \lim_{n\to\infty} \frac{2a^n}{2a^{n-1}} = a.$$

Suppose that $a > b$. Then

$$\ell = \lim_{n\to\infty} \frac{a^n\left(1 + \frac{b^n}{a^n}\right)}{a^{n-1}\left(1 + \frac{b^{n-1}}{a^{n-1}}\right)} = a \lim_{n\to\infty} \frac{1 + \left(\frac{b}{a}\right)^n}{1 + \left(\frac{b}{a}\right)^{n-1}} = a\frac{1}{1} = a.$$

By symmetry the limit will be b if $a < b$.
Summarising all cases in a single expression, $\ell = \max\{a, b\}$.

(ii) Since

$$\lim_{n\to\infty} \frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} = 1$$

the limit we want to calculate is an indeterminacy 1^∞. Hence

$$\lim_{n\to\infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n = e^c,$$

where

$$c = \lim_{n\to\infty} n\left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} - 1\right) = \lim_{n\to\infty} n\frac{\sqrt[n]{a} + \sqrt[n]{b} - 2}{2} = \frac{1}{2}\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1\right).$$

So we need to calculate

$$\ell(a) = \lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right)$$

and then $c = \frac{1}{2}[\ell(a) + \ell(b)]$. But

$$\ell(a) = \lim_{n\to\infty} \frac{a^{1/n} - 1}{1/n} = \lim_{n\to\infty} \frac{e^{(\log a/n)} - 1}{1/n} = \lim_{n\to\infty} \frac{e^{(\log a/n)} - 1}{\log a/n} \log a,$$

where we have first used the identity $a^x = e^{x\log a}$ and then multiplied and divided by $\log a$.
Denoting $\varepsilon_n = \log a/n$

$$\ell(a) = \log a \lim_{n\to\infty} \frac{e^{\varepsilon_n} - 1}{\varepsilon_n},$$

and since $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and we have the equivalence $e^{\varepsilon_n} - 1 \sim \varepsilon_n$ we conclude $\ell(a) = \log a$. Thus

$$c = \frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log\sqrt{ab}$$

and finally

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = e^{\log\sqrt{ab}} = \sqrt{ab}.$$

(iii) Using the identity $x^2 - y^2 = (x-y)(x+y)$ in the form $x - y = (x^2-y^2)/(x+y)$ (equivalently, multiplying and dividing by the conjugate of $x-y$),

$$\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right) = \lim_{n\to\infty} n\frac{\left(\sqrt{n^2+1}\right)^2 - n^2}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} n\frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n}.$$

Now,

$$\sqrt{n^2+1} = \sqrt{n^2\left(1+\frac{1}{n^2}\right)} = n\sqrt{1+\frac{1}{n^2}},$$

so

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}+\cancel{n}\cdot 1} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2}.$$

(iv) This time we need to use the identity $x^4 - y^4 = (x-y)(x^3+x^2y+xy^2+y^3)$ as $x - y = (x^4-y^4)/(x^3+x^2y+xy^2+y^3)$. Thus, if we denote

$$\ell = \lim_{n\to\infty}\sqrt{n}\left(\sqrt[4]{n^2+1}-\sqrt{n+1}\right),$$

then

$$\ell = \lim_{n\to\infty}\sqrt{n}\frac{n^2+1-(n+1)^2}{(n^2+1)^{3/4}+(n^2+1)^{1/2}(n+1)^{1/2}+(n^2+1)^{1/4}(n+1)+(n+1)^{3/2}}.$$

But $n^2+1-(n+1)^2 = \cancel{n^2}+\cancel{1}-\cancel{n^2}-2n-\cancel{1} = -2n$, and

$$(n^2+1)^{3/4} = n^{3/2}\left(1+\frac{1}{n^2}\right)^{3/4} \sim n^{3/2},$$

$$(n^2+1)^{1/2}(n+1)^{1/2} = n\left(1+\frac{1}{n^2}\right)^{1/2} n^{1/2}\left(1+\frac{1}{n}\right)^{1/2} \sim n^{3/2},$$

$$(n^2+1)^{1/4}(n+1) = n^{1/2}\left(1+\frac{1}{n^2}\right)^{1/4} n\left(1+\frac{1}{n}\right) \sim n^{3/2},$$

$$(n+1)^{3/2} = n^{3/2}\left(1+\frac{1}{n}\right)^{3/2} \sim n^{3/2},$$

so the denominator is $\sim 4n^{3/2}$. Accordingly,

$$\ell = \lim_{n\to\infty}\frac{-2n\sqrt{n}}{4n^{3/2}} = \lim_{n\to\infty}\frac{-2n^{3/2}}{4n^{3/2}} = -\frac{1}{2}.$$

(v) Factoring out the largest term of the numerator and of the denominator,

$$\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n} = \lim_{n\to\infty} \frac{3^{n+1}\left[1+(2/3)^{n+1}\right]}{3^n\left[1+(2/3)^n\right]} = \lim_{n\to\infty} 3\frac{1+(2/3)^{n+1}}{1+(2/3)^n} = 3.$$

(vi) This is a 1^∞ indeterminacy, so

$$\ell = \lim_{n\to\infty} \left(\frac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}} = e^c,$$

where

$$c = \lim_{n\to\infty} \frac{n^2-1}{2n}\left(\frac{n^2+1}{n^2-3n}-1\right) = \lim_{n\to\infty} \frac{n^2-1}{2n}\cdot\frac{\cancel{n^2}+1-\cancel{n^2}+3n}{n^2-3n} = \lim_{n\to\infty} \frac{n^2-1}{n^2-3n}\cdot\frac{1+3n}{2n} = \frac{3}{2}.$$

Thus $\ell = e^{3/2}$.

Problem 3.4

(i) Since $\sin n\pi = 0$ for every $n \in \mathbb{N}$ the sequence is identically 0, therefore the limit is 0.

(ii) We rewrite the expression as

$$\ell = \lim_{n\to\infty} \frac{n\left(e^{\frac{1}{n}}-e^{\sin\frac{1}{n}}\right)}{1-n\sin(1/n)} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}}-e^{\sin\frac{1}{n}}}{\frac{1}{n}-\sin\frac{1}{n}} = \lim_{n\to\infty} e^{\sin\frac{1}{n}}\cdot\frac{e^{\frac{1}{n}-\sin\frac{1}{n}}-1}{\frac{1}{n}-\sin\frac{1}{n}}.$$

If we now denote $\varepsilon_n = \frac{1}{n}-\sin\frac{1}{n} \xrightarrow[n\to\infty]{} 0$, then

$$\ell = \lim_{n\to\infty} e^{\sin\frac{1}{n}}\cdot\frac{e^{\varepsilon_n}-1}{\varepsilon_n} = 1,$$

because $e^{\varepsilon_n}-1 \sim \varepsilon_n$.

(iii) We can rewrite

$$\ell = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}},$$

and now apply Corollary 3.4.4 to Stolz's theorem to calculate the limit of a_n/a_{n-1} instead of the limit of $\sqrt[n]{a_n}$. Thus

$$\ell = \lim_{n\to\infty} \frac{n^n}{n!}\cdot\frac{(n-1)!}{(n-1)^{n-1}}.$$

But $n! = n\cdot(n-1)!$, so

$$\ell = \lim_{n\to\infty} \frac{n^n}{n(n-1)^{n-1}} = \lim_{n\to\infty} \frac{n^{n-1}}{(n-1)^{n-1}} = \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^{n-1} = \lim_{n\to\infty} \left(\frac{n-1+1}{n-1}\right)^{n-1}$$
$$= \lim_{n\to\infty} \left(1+\frac{1}{n-1}\right)^{n-1} = e.$$

Alternatively, we can use the equivalence $n! \sim (2\pi n)^{1/2}n^n e^{-n}$ and subtitute to obtain

$$\ell = \lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \lim_{n\to\infty} \frac{n}{\left((2\pi n)^{1/2}n^n e^{-n}\right)^{1/n}} = \lim_{n\to\infty} \frac{n}{(2\pi n)^{1/2n}ne^{-1}} = \lim_{n\to\infty} \frac{e}{(2\pi n)^{1/2n}} = e,$$

given that

$$\lim_{n\to\infty} (2\pi n)^{1/2n} = \lim_{n\to\infty} (\sqrt{2\pi})^{1/n}\left(\sqrt[n]{n}\right)^{1/2} = 1.$$

(iv)

$$\lim_{n\to\infty} n^{-3/n} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^{-3} = 1.$$

(v) The limit is 0 because $2^n \ll n!$; nevertheless we will prove the same limit in an alternative way, just for the sake of illustration. Let us expand

$$0 < \frac{2^n}{n!} = \frac{\overbrace{2\cdot 2\cdots 2\cdot 2\cdot 2}^{n \text{ times}}}{n\cdot(n-1)\cdots 3\cdot 2\cdot 1} = \frac{2}{n}\cdot\frac{2}{n-1}\cdots\frac{2}{3}\cdot\underbrace{\frac{2}{2}}_{=1}\cdot\underbrace{\frac{2}{1}}_{=2}.$$

Now, among all the fractions appearing in this expression (excluding the factors 1 and 2) the largest one is $2/3$. Therefore we obtain an upper bound if we replace all fractions by this one, i.e.,

$$0 < \frac{2^n}{n!} < \left(\frac{2}{3}\right)^{n-2} 2.$$

Since the rightmost side of this inequality goes to 0 as $n \to \infty$, by the sandwich rule

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

(vi) Likewise, this limit is 0 because $n^2 \ll 2^n$. But there is an alternative way to prove this. For that, we apply Stolz's theorem (the denominator is a monotonically increasing function that diverges to ∞) and calculate

$$\lim_{n\to\infty} \frac{n^2-(n-1)^2}{2^n-2^{n-1}} = \lim_{n\to\infty} \frac{2n-1}{2^{n-1}(2-1)} = \lim_{n\to\infty} \frac{2n-1}{2^{n-1}}.$$

And now apply Stolz's theorem again and calculate

$$\lim_{n\to\infty} \frac{2n-1-(2n-3)}{2^{n-1}-2^{n-2}} = \lim_{n\to\infty} \frac{2}{2^{n-2}(2-1)} = \lim_{n\to\infty} \frac{2}{2^{n-2}} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$$

(vii)

$$\lim_{n\to\infty} \frac{n^{n-1}}{(n-1)^n} = \lim_{n\to\infty} \frac{n^n}{n(n-1)^n} = \lim_{n\to\infty} \frac{1}{n}\cdot\left(\frac{n}{n-1}\right)^n = 0\cdot e = 0.$$

(viii) Applying Stolz's theorem

$$\ell = \lim_{n\to\infty} \frac{1+2\sqrt{2}+3\sqrt[3]{3}+\cdots+n\sqrt[n]{n}}{n^2}$$

(notice that the denominator is monotonically increasing and divergent) can be obtained as

$$\ell = \lim_{n\to\infty} \frac{n\sqrt[n]{n}}{n^2-(n-1)^2} = \lim_{n\to\infty} \frac{n\sqrt[n]{n}}{2n-1} = \lim_{n\to\infty} \frac{n}{2n-1}\cdot\sqrt[n]{n} = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

Problem 3.5

(i) This is a 1^∞ indeterminacy, thus

$$\ell = \lim_{n\to\infty}\left(\cos\frac{b}{n} + a\sin\frac{b}{n}\right)^n = e^c,$$

where

$$c = \lim_{n\to\infty} n\left(\cos\frac{b}{n} + a\sin\frac{b}{n} - 1\right) = \lim_{n\to\infty} n\left(\cos\frac{b}{n} - 1\right) + a\lim_{n\to\infty} n\sin\frac{b}{n}.$$

But $1-\cos\frac{b}{n}\sim\frac{b^2}{2n^2}$ and $\sin\frac{b}{n}\sim\frac{b}{n}$, thus

$$c = \lim_{n\to\infty} n\left(-\frac{b^2}{2n^2}\right) + a\lim_{n\to\infty} n\frac{b}{n} = 0 + ab = ab.$$

Therefore $\ell = e^{ab}$.

(ii) Here is another 1^∞ indeterminacy, so

$$\ell = \lim_{n\to\infty}\sqrt[u_n]{\frac{a-bu_n}{a+bu_n}} = e^c,$$

where

$$c = \lim_{n\to\infty}\frac{1}{u_n}\left(\frac{a-bu_n}{a+bu_n} - 1\right) = \lim_{n\to\infty}\frac{1}{u_n}\cdot\frac{\not{a} - bu_n - \not{a} - bu_n}{a+bu_n} = \lim_{n\to\infty}\frac{-2bu_n}{u_n(a+bu_n)}$$
$$= \lim_{n\to\infty}\frac{-2b}{a+bu_n} = -\frac{2b}{a}.$$

Therefore $\ell = e^{-2b/a}$.

Problem 3.6 We solve all these limits using Stolz's theorem.

(i) The denominator is $\log n$, a monotonically increasin sequence that diverges. Thus, if we call the limit ℓ,

$$\ell = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log n - \log(n-1)} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log\left(\frac{n}{n-1}\right)} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log\left(\frac{n-1+1}{n-1}\right)} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log\left(1+\frac{1}{n-1}\right)}$$
$$= \lim_{n\to\infty}\frac{\pi/n}{1/(n-1)} = \pi\lim_{n\to\infty}\frac{n-1}{n} = \pi.$$

(ii) If we denote the limit by ℓ, then

$$\log\ell = \lim_{n\to\infty}\log\left(\prod_{k=1}^{n}(2k-1)^{1/n^2}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\log\left((2k-1)^{1/n^2}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n^2}\log(2k-1)$$
$$= \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\log(2k-1)}{n^2}.$$

Now we apply Stolz's theorem and compute

$$\log\ell = \lim_{n\to\infty}\frac{\log(2n-1)}{n^2-(n-1)^2} = \lim_{n\to\infty}\frac{\log(2n-1)}{2n-1} = 0.$$

Therefore $\ell = 1$.

(iii) Taking the n^2 out of the sum

$$\ell = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2} \sin\frac{1}{k} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^2 \sin\frac{1}{k}}{n^2},$$

thus applying Stolz's theorem

$$\ell = \lim_{n\to\infty} \frac{n^2 \sin\frac{1}{n}}{n^2-(n-1)^2} = \lim_{n\to\infty} \frac{n^2\frac{1}{n}}{2n-1} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2},$$

where we have used $\sin\frac{1}{n} \sim \frac{1}{n}$.

Problem 3.7 If we denote ℓ the limit we want to calculate and apply Stolz's theorem (the denominator, $\log(n+1)$, is a monotonically and divergent sequence) we get

$$\ell = \lim_{n\to\infty} \frac{a_n/n}{\log(n+1)-\log n} = \lim_{n\to\infty} \frac{a_n}{n\log\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty} \frac{a_n}{n\log\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{a_n}{n\frac{1}{n}} = \lim_{n\to\infty} a_n = a,$$

using the equivalence $\log\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$.

Problem 3.8 The smallest term in the sum is $\frac{1}{\sqrt{n^2+3n}}$ (the one with the largest denominator) and the largest term is $\frac{1}{\sqrt{n^2+1}}$ (the one with the smallest denominator). Since there are $3n$ terms in the sum

$$\frac{3n}{\sqrt{n^2+3n}} < \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}} < \frac{3n}{\sqrt{n^2+1}}.$$

For the two bounding sequences we have

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2+3n}} = 3, \qquad \lim_{n\to\infty} \frac{3n}{\sqrt{n^2+1}} = 3,$$

therefore applying the sandwich rule we conclude

$$\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}} = 3.$$

Problem 3.9

(a) $\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n-n+n}{n} = \lim_{n\to\infty} \frac{a_n-n}{n} + 1 = 1$ because $\frac{a_n-n}{n} \sim \frac{\ell}{n}$.

(b) $\lim_{n\to\infty} n\log\left(\frac{a_n}{n}\right) = \lim_{n\to\infty} n\log\left(1+\frac{a_n-n}{n}\right) = \lim_{n\to\infty} n\cdot\frac{a_n-n}{n} = \lim_{n\to\infty}(a_n-n) = \ell$, where we have used the equivalence $\log(1+\varepsilon_n) \sim \varepsilon_n$ for any sequence $\varepsilon_n \xrightarrow[n\to\infty]{} 0$.

Problem 3.10 Assume that

$$\lim_{n\to\infty} \sqrt[n^2]{\frac{a_n^n}{a_1a_2\cdots a_n}} = x$$

Then

$$\log x = \lim_{n\to\infty} \frac{1}{n^2}\left(n\log a_n - \sum_{k=1}^{n} \log a_k\right).$$

Since n^2 is a monotonically increasing, divergent sequence we can apply Stolz's theorem with $b_n = n^2$ and

$$a_n = n\log a_n - \sum_{k=1}^{n} \log a_k,$$

and calculate instead

$$\log x = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

On the one hand $n^2 - (n-1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1$. On the other hand,

$$\begin{aligned} a_n - a_{n-1} &= n\log a_n - \sum_{k=1}^{n} \log a_k - (n-1)\log a_{n-1} + \sum_{k=1}^{n-1} \log a_k = n\log a_n - (n-1)\log a_{n-1} - \log a_n \\ &= (n-1)\log a_n - (n-1)\log a_{n-1} = (n-1)\log\left(\frac{a_n}{a_{n-1}}\right). \end{aligned}$$

Therefore

$$\log x = \lim_{n\to\infty} \frac{n-1}{2n-1} \log\left(\frac{a_n}{a_{n-1}}\right) = \frac{\log \ell}{2} = \log\sqrt{\ell} \qquad \Rightarrow \qquad x = \sqrt{\ell}.$$

Problem 3.11

(i) We can write the sequence as $x_{n+1} = \sqrt{2x_n}$, with $x_0 = 1$. In order to know whether the sequence is monotonically increasing or decreasing we need to assess the sign of

$$x_{n+1} - x_n = \sqrt{2x_n} - x_n = \frac{2x_n - x_n^2}{\sqrt{2x_n} + x_n} = \frac{(2 - x_n)x_n}{\sqrt{2x_n} + x_n}.$$

(We have used the identity $x - y = (x^2 - y^2)/(x + y)$.) The sequence is clearly posivite, because $x_0 = 1 > 0$ and $x_{n+1} = \sqrt{2x_n} > 0$ whenever $x_n > 0$. Thus the sign of $x_{n+1} - x_n$ will be the sign of the factor $2 - x_n$.

We are going to prove now that $x_n < 2$ for all $n \in \mathbb{N}$. First, $x_0 = 1 < 2$; second, if $x_n < 2$ then $x_{n+1} = \sqrt{2x_n} < \sqrt{2\cdot 2} = 2$. Hence it is proven by induction. But with this we have simultaneously proven

$$x_n < 2, \qquad x_{n+1} - x_n > 0,$$

so the sequence increases monotonically and is bounded from above —hence converges. The limit can be obtained by taking limits in the recurrence equation:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n},$$

and if we denote $\lim_{n\to\infty} x_n = x$, this equation becomes

$$x = \sqrt{2x} \qquad \Rightarrow \qquad x^2 = 2x \qquad \Rightarrow \qquad x(x-2) = 0.$$

Of the two solutions of this equation, $x = 0$ and $x = 2$, the latter has to be the solution because the sequence begins at $x_0 = 1$ and increases.

(ii) We describe de sequence as $x_{n+1} = \sqrt{2+x_n}$, with $x_0 = 0$, and proceed as in (i):

$$x_{n+1} - x_n = \sqrt{2+x_n} - x_n = \frac{2+x_n-x_n^2}{\sqrt{2+x_n}+x_n}.$$

The equation $x^2 - x - 2 = 0$ has two roots, namely $x = -1$ and $x = 2$, and therefore $2+x_n - x_n^2 = (x_n+1)(2-x_n)$. Thus,

$$x_{n+1} - x_n = \frac{(x_n+1)(2-x_n)}{\sqrt{2+x_n}+x_n}.$$

The sign of $x_{n+1} - x_n$ is the sign of $2 - x_n$ —because, as in (i), $x_n > 0$ for all $n \in \mathbb{N}$.
Let us now prove —using induction— that $x_n < 2$ for all $n \in \mathbb{N}$. Clearly $x_0 = 0 < 2$, and if $x_n < 2$ then $x_{n+1} = \sqrt{2+x_n} < \sqrt{2+2} = 2$. So we have just proven that

$$x_n < 2, \qquad x_{n+1} - x_n > 0,$$

and thefore the sequence converges. Denoting $\lim_{n\to\infty} x_n = x$ and taking limits in the recurrence equation

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n} \qquad \Rightarrow x = \sqrt{2+x} \qquad \Rightarrow x^2 = 2+x \qquad \Rightarrow x^2 - x - 2 = 0,$$

whose roots are $x = -1$ and $x = 2$. Of them, the latter is the limit because the whole sequence is positive.

(iii) The difference of two consecutive terms is

$$u_{n+1} - u_n = 3 + \frac{u_n}{2} - u_n = 3 - \frac{u_n}{2} = \frac{6-u_n}{2},$$

so its sign will depend on whether $u_n < 6$ or $u_n > 6$. Let us prove by induction that it is the former. To begin with $u_0 = 0 < 6$. Now let us assume that $u_n < 6$. Then

$$u_{n+1} = 3 + \frac{u_n}{2} < 3 + \frac{6}{2} = 3+3 = 6,$$

so $u_n < 6$ implies $u_{n+1} < 6$, and this completes the proof. Then we can conclude that $u_{n+1} - u_n > 0$, hence the sequence is monotonically increasing. On the other hand 6 is an upper bound, so it is convergent. To calculate the limit we need to take limits in both sides of the recurrence equation. If we denote $\lim_{n\to\infty} u_n = \ell$, then

$$\ell = 3 + \frac{\ell}{2} \qquad \Rightarrow \qquad \ell = 6.$$

(iv) In this case $u_1 = 3$ and $u_{n+1} = 3 + 2u_n > 2u_n$. In particular this implies $u_2 > 2u_1 = 2 \cdot 3$; also $u_3 > 2u_2 = 2^2 \cdot 3$; $u_4 > 2u_3 = 2^3 \cdot 3$; etc. In general

$$u_n > 2^{n-1} \cdot 3.$$

But $2^{n-1} \xrightarrow[n\to\infty]{} \infty$, so $\lim_{n\to\infty} u_n = \infty$.

(v) We calculate the difference

$$u_{n+1} - u_n = \frac{u_n^3+6}{7} - u_n = \frac{u_n^3 - 7u_n + 6}{7}.$$

But $x^3 - 7x + 6 = (x-1)(x^2+x-6) = (x-1)(x-2)(x+3)$, therefore

$$u_{n+1} - u_n = \frac{1}{7}(u_n-1)(u_n-2)(u_n+3).$$

(a) Since $u_0 = 1/2$ we have $0 < u_0 < 1$, so let us try to prove by induction that the whole sequence is in $(0,1)$. Assume $0 < u_n < 1$. Then

$$0 < u_n^3 < 1 \qquad \Rightarrow \qquad 0 < \frac{u_n^3+6}{7} < \frac{1+6}{7} = 1,$$

therefore $0 < u_{n+1} < 1$. And since the whole sequence is in $(0,1)$ we have

$$u_n - 1 < 0, \qquad u_n - 2 < 0, \qquad u_n + 3 > 0,$$

and therefore u_n is monotonically increasing and bounded from above by 1 —hence convergent. We can calculate the limit, ℓ, by solving the equation

$$\ell = \frac{\ell^3+6}{7} \qquad \Rightarrow \qquad \ell^3 - 7\ell + 6 = 0.$$

There are three solutions, namely $\ell = -3$, 1, and 2. Of those three the limit must be 1 because $0 < u_n < 1$, $u_0 = 1/2$, and u_n increases monotonically.

(b) Since $u_0 = 3/2$ we have $1 < u_0 < 2$, so let us try to prove by induction that the whole sequence is in $(1,2)$. Assume $1 < u_n < 2$. Then

$$1 < u_n^3 < 8 \qquad \Rightarrow \qquad \frac{1+7}{7} < \frac{u_n^3+6}{7} < \frac{8+6}{7} \qquad \Rightarrow \qquad 1 < \frac{u_n^3+6}{7} < 2,$$

therefore $1 < u_{n+1} < 2$. And since the whole sequence is in $(1,2)$ we have

$$u_n - 1 > 0, \qquad u_n - 2 < 0, \qquad u_n + 3 > 0,$$

and therefore u_n is monotonically decreasing and bounded from below by 1 —hence convergent. Again the limit can only be $\ell = -3$, 0, and 1. Of those three the limit must be 1 because $1 < u_n < 2$, $u_0 = 3/2$, and u_n decreases monotonically.

(c) Since $u_0 = 3$ we have $2 < u_0$, so let us try to prove by induction that the whole sequence is bounded from below by 2. Assume $u_n > 2$. Then

$$u_n^3 > 8 \qquad \Rightarrow \qquad \frac{u_n^3+6}{7} > \frac{8+6}{7} = 2,$$

therefore $u_{n+1} > 2$. This completes the proof. Now,

$$u_n - 1 > 0, \qquad u_n - 2 > 0, \qquad u_n + 3 > 0,$$

and therefore u_n is monotonically increasing —but there is no upper bound that we know of. Suppose there is such an upper bound —even though we do not know which one—; it that case the sequence would have a limit. But the limit can only be $\ell = -3$, 1, or 2. And neither of them can be because $u_0 = 3$ and the sequence is monotonically *increasing*. Therefore there is no upper bound for this sequence —which means that it *diverges*.

Problem 3.12

(a) Let us calculate the difference

$$a_{n+1} - a_n = \sqrt{1+3a_n} - 1 - a_n = \frac{1+3a_n-(1+a_n)^2}{\sqrt{1+3a_n}+1+a_n} = \frac{\cancel{1}+\cancel{3}a_n-\cancel{1}-\cancel{2a_n}-a_n^2}{\sqrt{1+3a_n}+1+a_n}$$
$$= \frac{a_n - a_n^2}{\sqrt{1+3a_n}+1+a_n} = \frac{a_n(1-a_n)}{\sqrt{1+3a_n}+1+a_n}.$$

The sequence is $a_n > 0$. The reason is that $a_0 = 1/2 > 0$ and if $a_n > 0$ then $a_{n+1} = \sqrt{1+3a_n} - 1 > \sqrt{1} - 1 = 0$. Accordingly both the denominator of the fraction and the first factor in the numerator are positive. On the other hand, $a_0 = 1/2 < 1$, and if $a_n < 1$ then $a_{n+1} = \sqrt{1+3a_n} - 1 < \sqrt{1+3} - 1 = 2 - 1 = 1$, so 1 is an upper bound for the sequence —which means that the second factor in the numerator is also positive. Hence $a_{n+1} - a_n > 0$ and the sequence is monotonically increasing. Since is is also bounded from above —by 1— it has a limit ℓ —to be determined.
If we now take limits in the recurrence equation we obtain

$$\begin{aligned} \ell = \sqrt{1+3\ell} - 1 \quad &\Rightarrow \quad \ell + 1 = \sqrt{1+3\ell} \quad \Rightarrow \quad (\ell+1)^2 = 1 + 3\ell \\ &\Rightarrow \quad \ell^2 + \not{2}\ell + \not{1} = \not{1} + \not{3}\ell \quad \Rightarrow \quad \ell^2 = \ell. \end{aligned}$$

The only solutions are $\ell = 0$ and 1, but since $a_0 = 1/2$ and a_n is monotonically increasing, the limit must be $\ell = 1$.

(b) Both the numerator and the denominator go to 0 as $n \to \infty$, so we face a $0/0$ indeterminacy. Substituting the recurrence, and using the identity $x - y = (x^2 - y^2)/(x+y)$,

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}-1}{a_n - 1} &= \lim_{n\to\infty} \frac{\sqrt{1+3a_n} - 2}{a_n - 1} = \lim_{n\to\infty} \frac{1 + 3a_n - 4}{(\sqrt{1+3a_n}+2)(a_n - 1)} \\ &= \lim_{n\to\infty} \frac{3a_n - 3}{(\sqrt{1+3a_n}+2)(a_n-1)} = \lim_{n\to\infty} \frac{3\cancel{(a_n-1)}}{(\sqrt{1+3a_n}+2)\cancel{(a_n-1)}} \\ &= \frac{3}{\sqrt{1+3}+2} = \frac{3}{4}. \end{aligned}$$

Problem 3.13

(a) We need to compute the product $(b_{n+1} - b_n)(b_n - b_{n-1})$ and prove that it is negative. In order to do that a good strategy is to write down this expression as a function of b_n alone. The term b_{n+1} is directly provided by the recurrence equation

$$b_{n+1} = 1 - \frac{b_n}{2}.$$

On the other hand, the same recurrence implies

$$b_n = 1 - \frac{b_{n-1}}{2} \quad \Rightarrow \quad b_{n-1} = 2(1 - b_n).$$

Therefore

$$(b_{n+1} - b_n)(b_n - b_{n-1}) = \left(1 - \frac{b_n}{2} - b_n\right)(b_n - 2 + 2b_n) = \frac{2 - 3b_n}{2}(3b_n - 2) = -\frac{(3b_n-2)^2}{2}.$$

The last expression is always negative unless $b_n = 2/3$. So as long as $b_n \neq 2/3$ the sequence keeps alternating. On the other hand, if $b_n = 2/3$ then

$$b_{n+1} = 1 - \frac{b_n}{2} = 1 - \frac{1}{3} = \frac{2}{3},$$

so the rest of the sequence remains fixed at $2/3$.

(b) If $\lim_{n\to\infty} b_n = \ell$ then, taking limits in the recurrence equation we obtain

$$\ell = 1 - \frac{\ell}{2} \quad \Rightarrow \quad \ell = \frac{2}{3}.$$

(c) Substituting $\ell = 2/3$,

$$b_{n+1} - \frac{2}{3} = 1 - \frac{b_n}{2} - \frac{2}{3} = \frac{1}{3} - \frac{b_n}{2} = \frac{1}{2}\left(\frac{2}{3} - b_n\right),$$

factoring out a $1/2$ factor. Taking absolute values in both sides of the obtained equation we get to the desired result.

(d) If $\left|b_n - \frac{2}{3}\right| = \frac{1}{2}\left|b_{n-1} - \frac{2}{3}\right|$, then, setting $n = 1$ we obtain

$$\left|b_1 - \frac{2}{3}\right| = \frac{1}{2}\left|b_0 - \frac{2}{3}\right| = \frac{1}{2}\cdot\frac{2}{3}.$$

Setting $n = 2$,

$$\left|b_2 - \frac{2}{3}\right| = \frac{1}{2}\left|b_1 - \frac{2}{3}\right| = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{2^2}\cdot\frac{2}{3}.$$

Setting $n = 3$,

$$\left|b_3 - \frac{2}{3}\right| = \frac{1}{2}\left|b_2 - \frac{2}{3}\right| = \frac{1}{2}\cdot\frac{1}{2^2}\cdot\frac{2}{3} = \frac{1}{2^3}\cdot\frac{2}{3}.$$

And we can continue this way to obtain the general expression

$$\left|b_n - \frac{2}{3}\right| = \frac{1}{2^n}\cdot\frac{2}{3}$$

valid for all $n \in \mathbb{N}$. If we take limits,

$$\lim_{n\to\infty}\left|b_n - \frac{2}{3}\right| = \frac{2}{3}\lim_{n\to\infty}\frac{1}{2^n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty}\left(b_n - \frac{2}{3}\right) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} b_n = \frac{2}{3}.$$

Problem 3.14

(a) Clearly $x_1 = 1 > 0$. Now, if we assume $x_n > 0$ then also $1 + x_n > 0$ and $1 + 2x_n > 0$; therefore

$$x_{n+1} = \frac{x_n(1+x_n)}{1+2x_n} > 0,$$

and the results is proven by induction.

(b) Let us calculate

$$x_{n+1} - x_n = \frac{x_n(1+x_n)}{1+2x_n} - x_n = \frac{\cancel{x_n} + \cancel{x_n^2} - \cancel{x_n} - \cancel{2}x_n^2}{1+2x_n} = -\frac{x_n^2}{1+2x_n} < 0$$

because the denominator is positive and $x_n^2 > 0$. Hence the sequence decreases monotonically.

(c) Denoning ℓ the limit of x_n and taking limits in the recurrence we obtain

$$\ell = \frac{\ell(1+\ell)}{1+2\ell} \quad \Rightarrow \quad \ell(1+2\ell) = \ell + \ell^2 \quad \Rightarrow \quad \cancel{\ell} + \cancel{2}\ell^2 = \cancel{\ell} + \cancel{\ell^2} \quad \Rightarrow \quad \ell = 0.$$