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Calculus I

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Unit 4. Series

Exercises

Problems

Problem 4.1 Determine the convergent character of the following series of nonnegative terms:

(i) $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)^n$;

(ii) $\sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}$;

(iii) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n^4+1}}$;

(iv) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$;

(v) $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2+n}$;

(vi) $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$;

(vii) $\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}$;

(viii) $\sum_{n=1}^{\infty}\frac{3n-1}{(\sqrt{2})^n}$;

(ix) $\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}$;

(x) $\sum_{n=1}^{\infty}(\sqrt[n]{n}-1)^n$;

(xi) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}3^{-n}$;

(xii) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$;

(xiii) $\sum_{n=2}^{\infty}\frac{n^2}{(\log n)^n}$;

(xiv) $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$;

(xv) $\sum_{n=2}^{\infty}\frac{1}{n^{\log n}}$;

(xvi) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}$;

(xvii) $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$;

(xviii) $\sum_{n=2}^{\infty}\left(\frac{n}{n-1}\right)^n$.

Problem 4.2 Prove that the series

$$\sum_{n=1}^{\infty}\left(\frac{a}{2n-1}-\frac{b}{2n+1}\right)$$

converges if, and only if, $a=b$, and in that case calculate its sum.

Problem 4.3 Discuss, depending on the value of the parameter a in the given range, whether the following series converge or diverge:

(i) $\sum_{n=1}^{\infty}n(1+a)^n e^{-na}$, for $a>-1$;

(ii) $\sum_{n=1}^{\infty}\frac{n^n}{a^n n!}$, for $a>0$;

(iii) $\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+a}}$, for any $a\in\mathbb{R}$;

(iv) $\sum_{n=1}^{\infty}\frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$, for $a\geqslant 0$.

Problem 4.4 Determine whether the following series are absolutely convergent, and if not, whether they converge conditionally:

(i) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$;

(ii) $\sum_{n=1}^{\infty}\sin\left(\pi n+\frac{1}{n}\right)$;

(iii) $\sum_{n=1}^{\infty}(-1)^n\left(\arctan\frac{1}{n}\right)^2$;

(iv) $\sum_{n=1}^{\infty}(-1)^n(\arctan n)^2$;

(v) $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n^2-1}-n\right)$;

(vi) $\sum_{n=1}^{\infty}(-1)^n\log\left(\frac{n}{n+1}\right)$;

(vii) $\sum_{n=1}^{\infty}(-1)^n\left(1-\cos\frac{1}{n}\right)$;

(viii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\log(e^n+e^{-n})}$.

Problem 4.5 Sum the following series:

(i) $\sum_{n=0}^{\infty} \frac{3^{n+1} - 2^{n-3}}{4^n}$;

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$;

(iii) $\sum_{n=0}^{\infty} \frac{4n+1}{3^n}$;

(iv) $\sum_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right)$;

(v) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}}$;

(vi) $\sum_{n=1}^{\infty} \log\left[\frac{n(n+2)}{(n+1)^2}\right]$;

(vii) $\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor}$, with $|xy| < 1$;

(viii) $\sum_{n=0}^{\infty} \frac{1}{2^n} \cos \frac{2\pi n}{3}$.

Problem 4.6 Obtain the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n^2 + 2n}$ by rewriting it as a telescoping series.
HINT: Expand the general term in elementary fractions.

Problem 4.7 Let $\mathscr{C}_0$ be a circle of radius r. Let $\mathscr{Q}_0$ be a square inscribed in $\mathscr{C}_0$. Let $\mathscr{C}_1$ be the circle inscribed in $\mathscr{Q}_0$, and $\mathscr{Q}_1$ a square inscribed in $\mathscr{C}_1$. Continue the process this way and obtain the sequence of circles $\{\mathscr{C}_n\}_{n=0}^{\infty}$ with radii $\{r_n\}_{n=0}^{\infty}$. What is the sum of the areas of these infinitely many circles?

Problem 4.8 Calculate $\lim_{n\to\infty} a_n$, where $a_n = \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\cdots\sqrt[2^n]{2}$.
HINT: Calculate the limit of $\log a_n$ first.

Problem 4.9 Let $b_0 \in \mathbb{Z}$, $b_n \in \{0, 1, 2, \ldots, 9\}$, for $n = 1, 2, \ldots$, and form the series

$$\sum_{n=0}^{\infty} \frac{b_n}{10^n}.$$

(a) Prove that this series converges.

(b) Discuss the meaning of this series and why it is so important.

(c) Calculate its sum for $b_n = 9$ for all $n \geqslant 0$.

(d) Calculate its sum if $b_n = 1$ for n even and $b_n = 2$ for n odd.

Problem 4.10

(a) Prove (graphically or otherwise) that the equation $\tan x = x$ has a solution $(2n-1)\frac{\pi}{2} < \lambda_n < (2n+1)\frac{\pi}{2}$ for every $n \in \mathbb{N}$.

(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$.

Problem 4.11

(a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ two convergent series of nonnegative terms. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n b_n} < \infty$.
HINT: Use the inequality $xy \leqslant (x^2 + y^2)/2$.

(b) As an application prove that if the series of nonnegative terms $\sum_{n=1}^{\infty} a_n < \infty$ then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty$.

Problem 4.12 Let $\{u_n\}_{n=1}^{\infty}$ be the sequence of all positive integers containing *no zeros* in their decimal expression.

(a) Prove that $\sum_{n=1}^{\infty} \frac{1}{u_n} < 90$.
HINT: Group all terms u_n with the same number of decimal digits.

(b) What can you say about the series $\sum_{n=1}^{\infty} \frac{1}{w_n}$, where $\{w_n\}_{n=1}^{\infty}$ is the sequence of all positive integers containing *at least one zero* in their decimal expression?

Problem 4.13 In a real *tour-de-force* we are going to calculate the —apparently impossible— sum of the conditionally convergent series

$$\sum_{n=1}^{\infty}(-1)^n \log\left(\frac{n}{n+1}\right).$$

We will do that in steps:

(a) Show that

$$2\cdot 4\cdot 6\cdots(2n)=n!2^n, \qquad 1\cdot 3\cdot 5\cdots(2n-1)=\frac{(2n)!}{n!2^n}.$$

(b) Use Stirling to prove

$$2\cdot 4\cdot 6\cdots(2n)\sim\sqrt{2\pi n}e^{-n}(2n)^n, \qquad 1\cdot 3\cdot 5\cdots(2n-1)\sim\sqrt{2}e^{-n}(2n)^n.$$

(c) Show that the partial sum S_{2k} of the series above can be written

$$S_{2k}=\log\left(\frac{2^2\cdot 4^2\cdot 6^2\cdots(2k)^2}{1\cdot 3^2\cdot 5^2\cdots(2k-1)^2(2k+1)}\right).$$

(d) Use the Stirling formulas derived above to calculate the limit of S_{2k} when $k\to\infty$. Why does this provide the answer to the problem?

Problem 4.14 Suppose a certain series can be written as

$$\sum_{n=1}^{\infty}(\alpha_0 u_n+\alpha_1 u_{n+1}+\alpha_2 u_{n+2}), \qquad \alpha_0+\alpha_1+\alpha_2=0.$$

(a) Rewrite the general term as an ordinary telescoping series and provide a formula for the sum.

(b) Apply this result to calculate the sum

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)(n+2)}.$$

(c) Do the same for the general case

$$\sum_{n=1}^{\infty}(\alpha_0 u_n+\alpha_1 u_{n+1}+\cdots+\alpha_k u_{n+k}), \qquad \sum_{j=0}^{k}\alpha_j=0.$$

HINT: In (a), add and substract $\alpha_0 u_{n+1}$ and replace $\alpha_2=-(\alpha_0+\alpha_1)$. Use a similar procedure in (c).