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Calculus I

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Unit 4. Series

Solutions

D.4 Series

Problem 4.1

(i) Converges according to the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} < 1.$$

(ii) Converges because $\frac{1}{(3n-1)^2} \sim \frac{1}{9n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

(iii) Converges because $\frac{1}{\sqrt{2n^4+1}} \sim \frac{1}{\sqrt{2}n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

(iv) Diverges because $\frac{1}{\sqrt{n(n+1)}} \sim \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(v) Since $|\sin n| \leqslant 1$ we can write

$$\frac{|\sin n|}{n^2+n} < \frac{1}{n^2+n}.$$

Since $\frac{1}{n^2+n} \sim \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} < \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2+n} < \infty$$

by the comparison test.

(vi) Converges because $\sin\left(\frac{1}{n^2}\right) \sim \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

(vii) Diverges because $\arcsin\left(\frac{1}{\sqrt{n}}\right) \sim \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$.

(viii) Converges according to the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{3n-1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} < 1.$$

Alternatively,

$$\sum_{n=1}^{\infty} \frac{3n-1}{(\sqrt{2})^n} = 3\sum_{n=1}^{\infty} n\left(\frac{1}{\sqrt{2}}\right)^n - \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n.$$

The first series is arithmetic-geometric and the second is geometric, both with argument $\frac{1}{\sqrt{2}} < 1$.

(ix) Coverges according to the quotient test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n^n}.$$

But $\frac{n!}{(n+1)!} = \frac{1}{n+1}$ and $\frac{3^n}{3^{n+1}} = \frac{1}{3}$, so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3(n+1)n^n} = \frac{1}{3}\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1.$$

(x) Converges according to the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\sqrt[n]{n} - 1\right) = 0 < 1.$$

(xi) Converges according to the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1.$$

(xii) Converges according to the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1.$$

(xiii) Converges according to the root test:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\log n} = 0 < 1.$$

(xiv) If we rewrite

$$\sqrt{n^2+1} - n = \frac{\cancel{n^2} + 1 - \cancel{n^2}}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \sim \frac{1}{2n},$$

we easily conclude that the series diverges because $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(xv) $\log n$ is an ever increasing function of n, so eventually $\log n > 2$. Then $n^{\log n} > n^2$ and therefore

$$\frac{1}{n^{\log n}} < \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ we conclude that the series converges by the comparison test.

(xvi) Let us apply Cauchy's condensation test: the convergence of the series will be equivalent to that of

$$\sum_{k=1}^{\infty} \frac{2^k}{(\log 2^k)^{\log 2^k}} = \sum_{k=1}^{\infty} \frac{2^k}{(k\log 2)^{k\log 2}} = \sum_{k=1}^{\infty} \underbrace{\frac{1}{k^{k\log 2}} \left(\frac{2}{(\log 2)^{\log 2}}\right)^k}_{=c_k}.$$

If we apply the root test,

$$\lim_{k\to\infty} \sqrt[k]{c_k} = \lim_{k\to\infty} \frac{1}{k^{\log 2}} \left(\frac{2}{(\log 2)^{\log 2}}\right) = 0,$$

so the series converges.

(xvii) Since

$$\lim_{n\to\infty} \sqrt[n]{n} = 1,$$

then

$$\frac{1}{n\sqrt[n]{n}} \sim \frac{1}{n}.$$

Therefore the series diverges because $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(xviii) Since

$$\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n=e,$$

the series diverges (in any convergent series the general term must tend to zero).

Problem 4.2 Reducing the general term to a unique fraction,

$$\frac{a}{2n-1}-\frac{b}{2n+1}=\frac{2(a-b)n+a+b}{4n^2-1}\sim\begin{cases}\dfrac{2(a-b)}{n}, & \text{if } a\neq b,\\[2mm] \dfrac{a+b}{n^2}, & \text{if } a=b.\end{cases}$$

Clearly the series converges if, and only if, $a=b$.

Now, in the case $a=b$, the series is

$$S=\sum_{n=1}^{\infty}\left(\frac{a}{2n-1}-\frac{a}{2n+1}\right)=\sum_{n=1}^{\infty}(u_n-u_{n+1}),\qquad u_n=\frac{a}{2n-1}.$$

Then

$$S=u_1-\lim_{n\to\infty}u_n=a.$$

Problem 4.3

(i) Applying the root test,

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{n}(1+a)e^{-a}=(1+a)e^{-a}.$$

But $e^a>1+a$ for all $a\neq 0$, so $(1+a)e^{-a}<1$ for all $a\neq 0$ and the series converges. For $a=0$ we have $(1+a)e^{-a}=1$ and the series becomes

$$\sum_{n=1}^{\infty}n=\infty.$$

(ii) Using the quotient test,

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}}=\lim_{n\to\infty}\frac{n^n}{a^n n!}\cdot\frac{a^{n-1}(n-1)!}{(n-1)^{n-1}}=\frac{1}{a}\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{n-1}=\frac{e}{a},$$

so the series converges for $a>e$ and diverges for $a<e$. What happens for $a=e$ must be decided with a different argument. But using Stirling,

$$\frac{n^n}{e^n n!}\sim\frac{n^n}{e^n\sqrt{2\pi n}n^n e^{-n}}\sim\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{n^{1/2}},$$

so the series diverges for $a=e$ because $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}=\infty$ for any $\alpha\leqslant 1$.

(iii) It seems we might use the quotient test, but

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}}=\lim_{n\to\infty}\frac{n!e^n}{n^{n+a}}\cdot\frac{(n-1)^{n-1+a}}{(n-1)!e^{n-1}}=e\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n-1+a}=e\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n+a-1}=ee^{-1}=1,$$

so it does not decide in this case. Let us use Stirling instead,

$$\frac{n!e^n}{n^{n+a}}\sim\frac{\sqrt{2\pi n}n^n e^{-n}e^n}{n^{n+a}}=\frac{\sqrt{2\pi n}}{n^a}=\sqrt{2\pi}\frac{1}{n^{a-1/2}},$$

so the series converges for $a-1/2>1$ (i.e. $a>3/2$) and diverges otherwise.

(iv) First of all, the series converges trivially for $a = 0$, because all terms are zero in this case. For $a > 0$ we have

$$a_n = \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} > 0,$$

so we can try the quotien test. This amounts to computing the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(1+a)(1+a^2)\cdots(1+a^n)(1+a^{n+1})} \frac{(1+a)(1+a^2)\cdots(1+a^n)}{a^n}$$
$$= \lim_{n\to\infty} \frac{a}{(1+a^{n+1})} = \begin{cases} a, & \text{if } a < 1, \\ \frac{1}{2}, & \text{if } a = 1, \\ 0, & \text{if } a > 1. \end{cases}$$

Whichever the case, this limit is always smaller than 1, hence the series coverges for any $a \geqslant 0$.

Problem 4.4

(i) The series does not converge conditionally because $\log n < n$, so $1/\log n > 1/n$, and the harmonic series diverges. Thus

$$\sum_{n=2}^{\infty} \frac{1}{\log n} = \infty$$

by the comparison test.
It does converge conditionally though, as the Leibniz's test proves: $1/\log n$ is a monotonically decreasing sequence that tends to zero.

(ii) Let us first expand

$$\sin\left(n\pi + \frac{1}{n}\right) = \underbrace{\sin n\pi}_{=0} \cos\frac{1}{n} + \underbrace{\cos n\pi}_{=(-1)^n} \sin\frac{1}{n} = (-1)^n \sin\frac{1}{n}.$$

Thus this is an alternating series of general term $\sin(1/n)$. It is not absolutely convergent because $\sin(1/n) \sim 1/n$ and the harmonic series diverges. It is conditionally convergent though, because $\sin(1/n)$ is monotonically decreasing toward zero (Leibniz's test).

(iii) Since

$$\left(\arctan\frac{1}{n}\right)^2 \sim \frac{1}{n^2},$$

the series converges absolutely.

(iv) Since

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

the series does not converge (not even conditionally).

(v) We can write

$$\sqrt{n^2-1} - n = \frac{\left(\sqrt{n^2-1}-n\right)\left(\sqrt{n^2-1}+n\right)}{\sqrt{n^2-1}+n} = \frac{n^2-1-n^2}{\sqrt{n^2-1}+n} = \frac{-1}{\sqrt{n^2-1}+n},$$

so

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n^2-1}-n\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n^2-1}+n}.$$

Now,

$$\frac{1}{\sqrt{n^2-1}+n}\sim\frac{1}{2n},$$

hence the series does not converge absolutely, but it does conditionally because the general term (without sign) is monotonically decreasing toward zero (Leibniz's test).

(vi) Let us rewrite

$$\sum_{n=1}^{\infty}(-1)^n\log\left(\frac{n}{n+1}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}\log\left(\frac{n+1}{n}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}\log\left(1+\frac{1}{n}\right).$$

As

$$\log\left(1+\frac{1}{n}\right)\sim\frac{1}{n}$$

the series does not converge absolutely, but it does conditionally because the general term (without sign) is monotonically decreasing toward zero (Leibniz's test).

(vii) We know that $1-\cos\varepsilon_n\sim\varepsilon_n^2/2$ for any vanishing sequence ε_n. Then,

$$1-\cos\frac{1}{n}\sim\frac{1}{2n^2}$$

and the series converges absolutely.

(viii) First of all,

$$e^n+e^{-n}\sim e^n \quad\Rightarrow\quad \log(e^n+e^-n)\sim\log(e^n)=n.$$

Therefore

$$\frac{1}{\log(e^n+e^-n)}\sim\frac{1}{n}$$

and the series does not converge absolutely because the harmonic series diverges. It doe conditionally though, because the general term (without sign) is monotonically decreasing toward zero (Leibniz's test).

Problem 4.5 There are only three types of series of which we know the sum: geometric, arithmetic-geometric, and telescoping series. So the point of this exercise is to identify these three types of series within the proposed ones.

(i) We can write

$$\frac{3^{n+1}-2^{n-3}}{4^n}=3\frac{3^n}{4^n}-\frac{1}{8}\frac{2^n}{4^n}=3\left(\frac{3}{4}\right)^n-\frac{1}{8}\left(\frac{1}{2}\right)^n.$$

Therefore

$$\sum_{n=0}^{\infty}\frac{3^{n+1}-2^{n-3}}{4^n}=3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n-\frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=3\frac{1}{1-3/4}-\frac{1}{8}\frac{1}{1-1/2}=12-\frac{1}{4}=\frac{47}{4}.$$

(ii) This is an arithmetic-geometric series of argument $1/2$, so

$$\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{1/2}{(1-1/2)^2}=2.$$

(iii) This one is the sum of an arithmetic-geometric and a plain geometric series of argument $1/3$, so

$$\sum_{n=0}^{\infty}\frac{4n+1}{3^n}=4\sum_{n=0}^{\infty}n\left(\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n=4\frac{1/3}{(1-1/3)^2}+\frac{1}{1-1/3}=3+\frac{3}{2}=\frac{9}{2}.$$

Notice that in the arithmetic-geometric series $\sum_{n=0}^{\infty}nx^n=\sum_{n=1}^{\infty}nx^n$ because the $n=0$ term is zero.

(iv) If we denote $u_n=\sqrt{n}-\sqrt{n+1}$, then

$$\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}=\sqrt{n}-\sqrt{n+1}-\left(\sqrt{n+1}-\sqrt{n+2}\right)=u_n-u_{n+1}.$$

Thus,

$$\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)=u_1-\lim_{n\to\infty}u_n.$$

Since $u_1=1-\sqrt{2}$ and

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right)=\lim_{n\to\infty}\frac{-1}{\sqrt{n}+\sqrt{n+1}}=0,$$

the sum

$$\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)=1-\sqrt{2}.$$

(v) We can split

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}=\frac{\cancel{\sqrt{n+1}}}{\sqrt{n\cancel{(n+1)}}}-\frac{\cancel{\sqrt{n}}}{\sqrt{\cancel{n}(n+1)}}=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}=u_n-u_{n+1},\qquad u_n=\frac{1}{\sqrt{n}}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}=u_1-\lim_{n\to\infty}u_n=1.$$

(vi) Splitting

$$\log\left[\frac{n(n+2)}{(n+1)^2}\right]=\log\left(\frac{n}{n+1}\right)+\log\left(\frac{n+2}{n+1}\right)=\log\left(\frac{n}{n+1}\right)-\log\left(\frac{n+1}{n+2}\right)=u_n-u_{n+1},$$

where we can identify

$$u_n=\log\left(\frac{n}{n+1}\right),$$

we obtain

$$\sum_{n=1}^{\infty}\log\left[\frac{n(n+2)}{(n+1)^2}\right]=u_1-\lim_{n\to\infty}u_n=-\log 2.$$

(vii) We can split the terms of the series into even and odd:

$$\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor} = \sum_{k=0}^{\infty} x^{\lfloor \frac{2k}{2} \rfloor} y^{\lfloor \frac{2k+1}{2} \rfloor} + \sum_{k=0}^{\infty} x^{\lfloor \frac{2k+1}{2} \rfloor} y^{\lfloor \frac{2k+2}{2} \rfloor} = \sum_{k=0}^{\infty} x^k y^k + \sum_{k=0}^{\infty} x^k y^{k+1}$$
$$= \sum_{k=0}^{\infty} (xy)^k + y \sum_{k=0}^{\infty} (xy)^k = (1+y) \sum_{k=0}^{\infty} (xy)^k.$$

Since $|xy| < 1$, then

$$\sum_{k=0}^{\infty} (xy)^k = \frac{1}{1-xy}.$$

Therefore

$$\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor} = \frac{1+y}{1-xy}.$$

(viii) If we give values to n and evaluate $\cos(2\pi n/3)$ we observe a repetitive pattern:

n	0	1	2	3	4	5	6	7	8	$\cdots$
$\cos(2\pi n/3)$	1	$-1/2$	$-1/2$	1	$-1/2$	$-1/2$	1	$-1/2$	$-1/2$	$\cdots$

If $n = 3k$ then $\cos(2\pi n/3) = 1$, otherwise $\cos(2\pi n/3) = -1/2$. Therefore

$$S = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos\frac{2\pi n}{3} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{2^{3k}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{8^k},$$

where we have summed up for all n as if $\cos(2\pi n/3) = -1/2$ always, and then we have corrected for the multiples of 3 so that the contribution of each one is $1(= -1/2 + 3/2)$. Then

$$S = -\frac{1}{2} \frac{1}{1-\frac{1}{2}} + \frac{3}{2} \frac{1}{1-\frac{1}{8}} = -1 + \frac{12}{7} = \frac{5}{7}.$$

Problem 4.6 The denominator factorises as $n(n+1)(n+2)$, and the elementary fractions expansion yields

$$\frac{1}{n^3+3n^2+2n} = \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+1} + \frac{1}{n+2}\right)$$
$$= \underbrace{\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{=u_n} - \underbrace{\frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}_{=u_{n+1}}.$$

Since

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0,$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2+2n} = u_1 = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}.$$

Problem 4.7 Let r_n be the radius of the circle $\mathscr{C}_n$. The diagonal of $\mathscr{Q}_n$, the inscribed square will be $2r_n$, therefore its side will be $\sqrt{2}r_n$. This side is the diameter of $\mathscr{C}_{n+1}$, the circle inscribed in $\mathscr{Q}_n$, therefore

$$r_{n+1} = \frac{r_n}{\sqrt{2}}, \qquad r_0 = r,$$

and the areas of $\mathscr{C}_{n+1}$ and $\mathscr{C}_n$ will be related by

$$A_{n+1} = \pi r_{n+1}^2 = \pi \frac{r_n^2}{2} = \frac{A_n}{2}, \qquad A_0 = \pi r^2.$$

Solving this iterative equation is very easy, because applying successively the iteration,

$$A_1 = \frac{\pi r^2}{2}, \quad A_2 = \frac{\pi r^2}{4}, \quad A_3 = \frac{\pi r^2}{8}, \quad \cdots \quad A_n = \frac{\pi r^2}{2^n}, \quad \cdots$$

Thus,

$$\sum_{n=0}^{\infty} A_n = \pi r^2 \sum_{n=0}^{\infty} \frac{1}{2^n} = \pi r^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \pi r^2 \frac{1}{1-1/2} = 2\pi r^2.$$

Problem 4.8 Taking the logarithm of the sequence

$$\log a_n = \frac{1}{2}\log 2 + \frac{1}{4}\log 2 + \frac{1}{8}\log 2 + \cdots + \frac{1}{2^n}\log 2 = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}\right)\log 2,$$

therefore

$$\lim_{n\to\infty} \log a_n = \log 2 \sum_{n=1}^{\infty} \frac{1}{2^n} = \log 2 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \log 2 \frac{\frac{1}{2}}{1-\frac{1}{2}} = \log 2,$$

which implies $\lim_{n\to\infty} a_n = 2$.

Problem 4.9

(a) The series can be written as

$$\sum_{n=0}^{\infty} \frac{b_n}{10^n} = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{10^n},$$

so we will prove the convergence of the series on the right-hand side. The reason to separate out the term b_0 is because its nature is different from the rest of the coefficientes b_n.
Now, this is a series of nonnegative terms and $b_n \leqslant 9$ for all $n \in \mathbb{N}$, so

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} \leqslant \sum_{n=1}^{\infty} \frac{9}{10^n} = 9\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 9 \cdot \frac{1/10}{1-1/10} = 1.$$

The series converges by the comparison test. Not just that: its maximum value is 1, and it is reached when all $b_n = 9$.

(b) Take any real number, say $\pi = 3.141592653\dots$. Its decimal expression is an integer number followed by a decimal point and an infinite sequence of digits (numbers in the set $\{0,1,2,\dots,9\}$). The meaning of this positional expression is this:

$$\pi = 3 + 1\times 0.1 + 4\times 0.01 + 1\times 0.001 + 5\times 0.0001 + \cdots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \cdots$$

In other words, the decimal expression of any real number is a series of the form of the one we are discussing. The fact that the series converges means that this representation is meaningful, and not just a formal description of the number. As a matter of fact, the proof that the series starting in $n = 1$ can be at most 1 means that the decimal part of the real number is always in the interval $[0,1]$, as it should.

(c) This question is asking for the value of the real number 9.999999... Written as a series,

$$9.999999\cdots = \sum_{n=0}^{\infty} \frac{9}{10^n} = 9 \cdot \frac{1}{1-1/10} = 10.$$

So $9.999999\cdots = 10$.

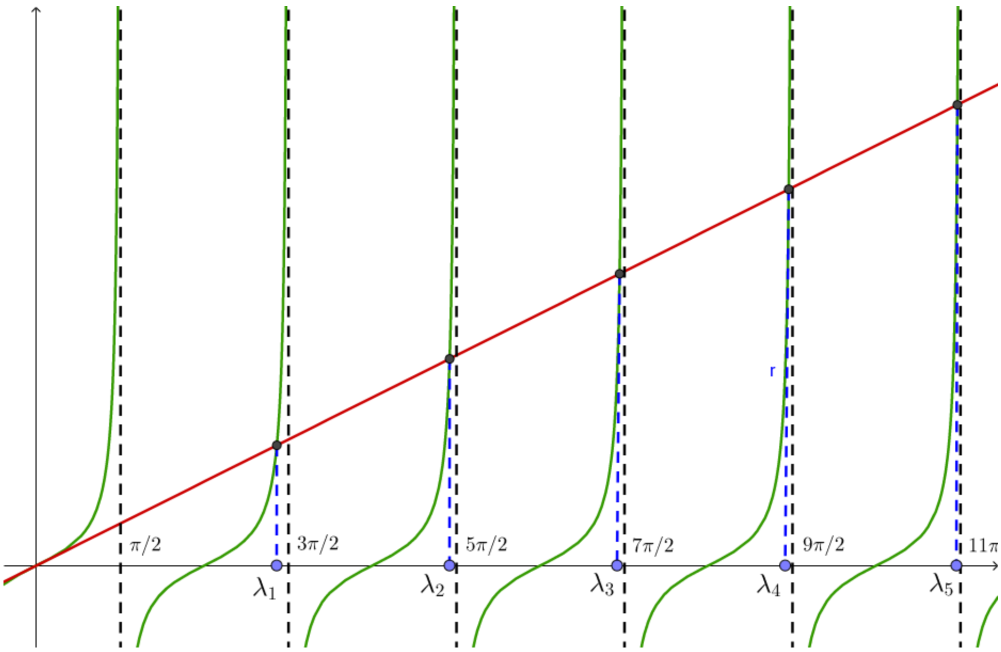
(d) Now we have to calculate the number 1.21212121...:

$$1.21212121\ldots = \sum_{n=0}^{\infty} \frac{b_n}{10^n} = \sum_{k=0}^{\infty} \frac{b_{2k}}{10^{2k}} + \sum_{k=0}^{\infty} \frac{b_{2k+1}}{10^{2k+1}} = \sum_{k=0}^{\infty} \frac{1}{10^{2k}} + \sum_{k=0}^{\infty} \frac{2}{10^{2k+1}}$$
$$= \sum_{k=0}^{\infty} \left(\frac{1}{100}\right)^k + \frac{2}{10} \sum_{k=0}^{\infty} \left(\frac{1}{100}\right)^k = \left(1 + \frac{1}{5}\right) \frac{1}{1-1/100} = \frac{6}{5} \cdot \frac{100}{99} = \frac{120}{99}.$$

In other words, $1.21212121\cdots = 120/99$, a rational number (its decimal expression is periodic).

Problem 4.10

(a) The location of the points λ_n is illustrated in the following figure, which proves graphically the result:



(b) Since $\lambda_n > (2n-1)\pi/2$, then $\lambda_n^{-2} < 4/\pi^2(2n-1)^2$. But

$$\frac{4}{\pi^2(2n-1)^2} \sim \frac{1}{\pi^2} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so the series $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$ by the comparison test.

Problem 4.11

(a) The inequality $xy \leqslant (x^2+y^2)/2$ implies

$$\sqrt{a_n b_n} = \sqrt{a_n}\sqrt{b_n} \leqslant \frac{a_n + b_n}{2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{a_n + b_n}{2} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\right) < \infty$$

because the two series of the right-hand side converge. Therefore the series $\sum_{n=1}^{\infty} \sqrt{a_n b_n} < \infty$ by the comparison test.

(b) If we take $b_n = 1/n^2$, the result is a straightforward application of the previous result because $\sum_{n=1}^{\infty} n^{-2} < \infty$.

Problem 4.12

(a) Let $\mathscr{U}_k$ denote the set of positive integers with exactly k digits, none of which is zero. Clearly,

$$\sum_{n=1}^{\infty} \frac{1}{u_n} = \sum_{k=1}^{\infty} \sum_{u \in \mathscr{U}_k} \frac{1}{u},$$

which amounts to nothing but performing the sum grouping those terms corresponding to integers with the same number of digits. The advantage of doing this is that we know that the smallest integer in $\mathscr{U}_k$ will have all its digits equal to 1, and its largest all its digits equal to 9. Now, the smallest integer in $\mathscr{U}_k$ satisfies

$$\min \mathscr{U}_k = \underbrace{111\cdots1}_{k \text{ digits}} > 1\underbrace{00\cdots0}_{k-1 \text{ zeros}} = 10^{k-1},$$

therefore $u > 10^{k-1}$ for all $u \in \mathscr{U}_k$ and

$$\frac{1}{u} < \frac{1}{10^{k-1}}.$$

As there are 9^k integers in $\mathscr{U}_k$ (each of the k digits can be anything between 1 and 9), then

$$\sum_{u \in \mathscr{U}_k} \frac{1}{u} < \sum_{u \in \mathscr{U}_k} \frac{1}{10^{k-1}} = \frac{9^k}{10^{k-1}} = 9\left(\frac{9}{10}\right)^{k-1}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{u_n} < 9\sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^{k-1} = 9\sum_{k=0}^{\infty} \left(\frac{9}{10}\right)^{k} = 9\frac{1}{1-9/10} = 90.$$

(b) We know that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{u_n}}_{\text{integers with no zeros}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{w_n}}_{\text{integers with zeros}}.$$

But the series of the left-hand side is the divergent harmonic series, and we have proven that the first series of the right-hand side is convergent, so the second one must be divergent.

Problem 4.13

(a) If we write $2 \cdot 4 \cdot 6 \cdots (2n) = (2 \cdot 1)(2 \cdot 2)(2 \cdot 3) \cdots (2 \cdot n)$ we see that there are n factors 2 in the product, and the remaining factors form the product $1 \cdot 2 \cdot 3 \cdots n = n!$, hence the result. As for the product $1 \cdot 3 \cdot 5 \cdots (2n-1)$, we need to realise that

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1) \cdot (2n) = 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2 \cdot 4 \cdot 6 \cdots (2n);$$

hence

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{(2n)!}{n!2^n}.$$

(b) On the one hand,

$$2 \cdot 4 \cdot 6 \cdots (2n) = n!2^n \sim \sqrt{2\pi n}e^{-n}n^n 2^n = \sqrt{2\pi n}e^{-n}(2n)^n,$$

and on the other hand,

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{n!2^n} \sim \frac{\sqrt{4\pi n}e^{-2n}(2n)^{2n}}{\sqrt{2\pi n}e^{-n}(2n)^n} = \sqrt{2}e^{-n}(2n)^n.$$

(c) Let us calculate a few terms:

$$S_2 = -\log\frac{1}{2} + \log\frac{2}{3} = \log\left(\frac{2^2}{1 \cdot 3}\right),$$
$$S_4 = -\log\frac{1}{2} + \log\frac{2}{3} - \log\frac{3}{4} + \frac{4}{5} = \log\left(\frac{2^2 \cdot 4^2}{1 \cdot 3^2 \cdot 5}\right),$$
$$S_6 = -\log\frac{1}{2} + \log\frac{2}{3} - \log\frac{3}{4} + \frac{4}{5} - \log\frac{5}{6} + \log\frac{6}{7} = \log\left(\frac{2^2 \cdot 4^2 \cdot 6^2}{1 \cdot 3^2 \cdot 5^2 \cdot 7}\right),$$

so the pattern seems to hold. Let us prove the result by induction. Assume

$$S_{2k} = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)}\right).$$

We must prove that

$$S_{2(k+1)} = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2(2k+2)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)^2(2k+3)}\right).$$

In order to do it we notice that

$$\begin{aligned} S_{2(k+1)} =& S_{2k} - \log\left(\frac{2k+1}{2k+2}\right) + \log\left(\frac{2k+2}{2k+3}\right) = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)}\right) \\ &+ \log\left(\frac{(2k+2)^2}{(2k+1)(2k+3)}\right) = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2(2k+2)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)^2(2k+3)}\right), \end{aligned}$$

which is what we wanted to prove.

(d) We have

$$2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2 \sim 2\pi k e^{-2k}(2k)^{2k}, \qquad 1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2 \sim 2e^{-2k}(2k)^{2k},$$

therefore

$$\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)} \sim \frac{\cancel{2}\pi k \cancel{e^{-2k}}\,\cancel{(2k)^{2k}}}{\cancel{2}\cancel{e^{-2k}}\,\cancel{(2k)^{2k}}} = \pi k,$$

so

$$S_{2k} \sim \log\left(\frac{\pi k}{2k+1}\right)$$

and therefore

$$\lim_{k\to\infty} S_{2k} = \log\left(\frac{\pi}{2}\right).$$

$\{S_{2k}\}_{k=1}^{\infty}$ is a subsequence of $\{S_k\}_{k=1}^{\infty}$. Since the latter converges the limit of the former must be the same. Hence

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty}(-1)^n \log\left(\frac{n}{n+1}\right) = \log\left(\frac{\pi}{2}\right).$$

Problem 4.14

(a) Following the hint, we can rewrite the general term of the series as

$$\begin{aligned} a_n &= \alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2} = \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} - \alpha_0 u_{n+1} - (\alpha_0+\alpha_1)u_{n+2} \\ &= \left[\alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1}\right] - \left[\alpha_0 u_{n+1} + (\alpha_0+\alpha_1)u_{n+2}\right] = U_n - U_{n+1}, \end{aligned}$$

where $U_n \equiv \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1}$. Thus

$$\sum_{n=1}^{\infty}(\alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2}) = \alpha_0 u_1 + (\alpha_0+\alpha_1)u_2 - \lim_{n\to\infty}\left[\alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1}\right].$$

(b) Expanding the rational expression into elementary fractions we find

$$\frac{2n+1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2}\right),$$

so $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = -3$ and $u_n = 1/n$. Therefore

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)(n+2)} = \frac{1}{2}(u_1 + 3u_2 - 0) = \frac{5}{4}.$$

(c) In the general case we can rewrite the general term as

$$\begin{aligned} a_n &= \alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2} + \cdots + \alpha_{k-1}u_{n+k-1} + \alpha_k u_{n+k} \\ &= \alpha_0 u_n + (\boxed{\alpha_0} + \alpha_1)u_{n+1} + \alpha_2 u_{n+2} + \cdots + \alpha_{k-1}u_{n+k-1} + \alpha_k u_{n+k} \\ &\quad - \boxed{\alpha_0 u_{n+1}} \\ &= \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} + (\boxed{\alpha_0+\alpha_1} + \alpha_2)u_{n+2} + \cdots + \alpha_{k-1}u_{n+k-1} + \alpha_k u_{n+k} \\ &\quad - \alpha_0 u_{n+1} - \boxed{(\alpha_0+\alpha_1)u_{n+2}} \\ &= \cdots = \\ &= \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} + (\alpha_0+\alpha_1+\alpha_2)u_{n+2} + \cdots + (\boxed{\alpha_0+\cdots+\alpha_{k-2}} + \alpha_{k-1})u_{n+k-1} + \alpha_k u_{n+k} \\ &\quad - \alpha_0 u_{n+1} - (\alpha_0+\alpha_1)u_{n+2} - \cdots - \boxed{(\alpha_0+\alpha_1+\cdots+\alpha_{k-2})u_{n+k-1}}, \end{aligned}$$

and since $\alpha_k u_{n+k} = -(\alpha_0 + \cdots + \alpha_{k-2} + \alpha_{k-1})u_{n+k}$, we finally have $a_n = U_n - U_{n+1}$, where

$$U_n = \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} + (\alpha_0+\alpha_1+\alpha_2)u_{n+2} + \cdots + (\alpha_0+\cdots+\alpha_{k-2}+\alpha_{k-1})u_{n+k-1}.$$