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Calculus I

Pablo Catalán Fernández y José A. Cuesta Ruiz

Unit 5. Limit of a Function

Solutions

D.5 Limit of a Function

Problem 5.1

(i) We use the identity $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$ and obtain

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{\cancel{x-a}} = na^{n-1}.$$

(ii) We use the identity $x - a = (\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})$ and get

$$\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a} = \lim_{x\to a} \frac{\cancel{\sqrt{x} - \sqrt{a}}}{\cancel{(\sqrt{x} - \sqrt{a})}(\sqrt{x} + \sqrt{a})} = \frac{1}{2\sqrt{a}}.$$

(iii) Here we need to use two identities. Since $64 = 8^2 = 4^3$,

$$x - 64 = (\sqrt{x} - 8)(\sqrt{x} + 8), \qquad x - 64 = (\sqrt[3]{x} - 4)\left(\sqrt[3]{x^2} + 4\sqrt[3]{x} + 16\right).$$

Then,

$$\lim_{x\to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x} - 4} = \lim_{x\to 64} \frac{\cancel{(x-64)}\left(\sqrt[3]{x^2} + 4\sqrt[3]{x} + 16\right)}{\cancel{(x-64)}(\sqrt{x} + 8)}$$

(iv) We can rewrite

$$1 - \sqrt{1-x^2} = \frac{\left(1 - \sqrt{1-x^2}\right)\left(1 + \sqrt{1-x^2}\right)}{1 + \sqrt{1-x^2}} = \frac{1 - (1-x^2)}{1 + \sqrt{1-x^2}} = \frac{x^2}{1 + \sqrt{1-x^2}}.$$

Therefore

$$\lim_{x\to 0} \frac{1 - \sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}\left(1 + \sqrt{1-x^2}\right)} = \lim_{x\to 0} \frac{1}{1 + \sqrt{1-x^2}} = \frac{1}{2}.$$

(v) We can rewrite

$$\frac{1}{(1-x)^3} - 1 = \frac{1 - (1-x)^3}{(1-x)^3} = \frac{1 - (1 - 3x + 3x^2 - x^3)}{(1-x)^3} = \frac{3x - 3x^2 + x^3}{(1-x)^3} = \frac{x(3 - 3x + x^2)}{(1-x)^3}.$$

Thus

$$\lim_{x\to 0} \frac{\frac{1}{(1-x)^3} - 1}{x} = \lim_{x\to 0} \frac{\cancel{x}(3 - 3x + x^2)}{\cancel{x}(1-x)^3} = \lim_{x\to 0} \frac{3 - 3x + x^2}{(1-x)^3} = 3.$$

(vi) We can rewrite

$$\frac{1}{\sqrt{x} - 1} = \frac{\sqrt{x} + 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \frac{\sqrt{x} + 1}{x - 1}.$$

Therefore

$$\lim_{x\to 1} \left(\frac{1}{\sqrt{x} - 1} - \frac{2}{x-1}\right) = \lim_{x\to 1} \left(\frac{\sqrt{x} + 1}{x-1} - \frac{2}{x-1}\right) = \lim_{x\to 1} \frac{\sqrt{x} + 1 - 2}{x-1} = \lim_{x\to 1} \frac{\sqrt{x} - 1}{x-1}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{(\sqrt{x} + 1)\cancel{(x-1)}} = \lim_{x\to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$

Problem 5.2

(i) If $x \to 0$ we know that $\sin 2x^3 \sim 2x^3$, so

$$\lim_{x\to 0} \frac{(\sin 2x^3)^2}{x^6} = \lim_{x\to 0} \frac{4\cancel{x^6}}{\cancel{x^6}} = 4.$$

(ii) We can divide numerator and denominator by x:

$$\ell = \lim_{x\to 0} \frac{\frac{\tan x^2}{x} + 2}{1+x} = \lim_{x\to 0} \frac{x\frac{\tan x^2}{x^2} + 2}{1+x}.$$

Now,

$$\lim_{x\to 0} \frac{\tan x^2}{x^2} = \lim_{x\to 0} \frac{1}{\cos x^2}\frac{\sin x^2}{x^2} = 1 \qquad \Rightarrow \qquad \lim_{x\to 0} x\frac{\tan x^2}{x^2} = 0.$$

Thus $\ell = 2$.

(iii) Expanding $\sin(x+a) = \sin x \cos a + \cos x \sin a$,

$$\begin{aligned} \ell &= \lim_{x\to 0} \frac{\sin(x+a) - \sin a}{x} = \lim_{x\to 0} \frac{\sin x \cos a - \sin a(1-\cos x)}{x} \\ &= \cos a \lim_{x\to 0} \frac{\sin x}{x} - \sin a \lim_{x\to 0} \frac{1-\cos x}{x} = \cos a \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} - \sin a \underbrace{\lim_{x\to 0} \frac{x^2/2}{x}}_{=0} = \cos a, \end{aligned}$$

since $\sin x \sim x$ and $1-\cos x \sim x^2/2$ when $x \to 0$.

(iv) This limit is an indeterminacy 1^∞, therefore

$$\ell = \lim_{x\to 0}(1+x)^{1/x} = e^c, \qquad c = \lim_{x\to 0} \frac{1}{x}(1+x-1) = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} = 1.$$

Thus $\ell = e$.

(v) Since $\log(1-2x) \sim -2x$ and $\sin x \sim x$ when $x \to 0$,

$$\lim_{x\to 0} \frac{\log(1-2x)}{\sin x} = \lim_{x\to 0} \frac{-2\cancel{x}}{\cancel{x}} = -2.$$

(vi) This limit is an indeterminacy 1^∞, therefore

$$\ell = \lim_{x\to 0}(1+\sin x)^{2/x} = e^c, \qquad c = \lim_{x\to 0} \frac{2}{x}(1+\sin x - 1) = \lim_{x\to 0} \frac{2\sin x}{x} = 2.$$

Thus $\ell = e^2$.

(vii) We can factor out $e^{\sin x}$ in the numerator to obtain

$$\ell = \lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0} e^{\sin x}\frac{e^{x-\sin x} - 1}{x - \sin x} = \lim_{x\to 0} \frac{e^{x-\sin x} - 1}{x - \sin x},$$

because $\lim_{x\to 0} e^{\sin x} = 1$. Now $x - \sin x \to 0$ as $x \to 0$, therefore $e^{x-\sin x} - 1 \sim x - \sin x$, so

$$\ell = \lim_{x\to 0} \frac{\cancel{x - \sin x}}{\cancel{x - \sin x}} = 1.$$

(viii) We can rewrite

$$\frac{\tan x - \sin x}{x^3} = \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x^2} \cdot \frac{1}{\cos x}.$$

But $\sin x \sim x$ and $1 - \cos x \sim x^2/2$ as $x \to 0$, so

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} \cdot \lim_{x\to 0} \frac{\cancel{x^2}/2}{\cancel{x^2}} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

(ix) First of all,

$$\lim_{x\to 0} \frac{x}{\sin x} = 1, \qquad \lim_{x\to 0} \frac{\sin x}{\sin x - x} = \lim_{x\to 0} \frac{\frac{\sin x}{x}}{\frac{\sin x}{x} - 1} = \infty,$$

so the limit is an indeterminacy 1^∞. Thus,

$$\ell = \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}} = e^c, \qquad c = \lim_{x\to 0} \frac{\sin x}{\sin x - x} \left(\frac{x}{\sin x} - 1\right) = \lim_{x\to 0} \frac{\sin x}{\sin x - x} \cdot \frac{x - \sin x}{\sin x} = -1.$$

Therefore $\ell = 1/e$.

(x) Another indeterminacy 1^∞, so

$$\lim_{x\to 0} (\cos x)^{1/x^2} = e^c, \qquad c = \lim_{x\to 0} \frac{1}{x^2}(\cos x - 1) = \lim_{x\to 0} \frac{-\cancel{x^2}/2}{\cancel{x^2}} = -1/2.$$

Therefore $\ell = 1/\sqrt{e}$.

(xi) The best strategy here is to change the variable to $t = x - \pi$, so that $x \to \pi$ becomes $t \to 0$. Then

$$\sin(x/2) = \sin(\pi/2 + t/2) = \underbrace{\sin(\pi/2)}_{=1} \cos(t/2) + \underbrace{\cos(\pi/2)}_{=0} \sin(t/2) = \cos(t/2).$$

Then

$$\lim_{x\to \pi} \frac{1 - \sin(x/2)}{(x-\pi)^2} = \lim_{t\to 0} \frac{1 - \cos(t/2)}{t^2} = \lim_{t\to 0} \frac{\cancel{t^2}/8}{\cancel{t^2}} = \frac{1}{8},$$

where we have made use of the equivalence, valid for $t \to 0$,

$$1 - \cos(t/2) \sim \frac{(t/2)^2}{2} = \frac{t^2}{8}.$$

(xii) We first need to manipulate a little this expression. For that purpose we substract and add 1 to the numerator to write

$$\ell = \lim_{x\to 0} \frac{a^x - b^x}{x} = \lim_{x\to 0} \frac{a^x - 1 - b^x + 1}{x} = \lim_{x\to 0} \frac{a^x - 1}{x} - \lim_{x\to 0} \frac{b^x - 1}{x}.$$

We can calculate separately

$$\ell_a = \lim_{x\to 0} \frac{a^x - 1}{x}.$$

Whatever result is yields, the other limit will be the same replacing a by b. But first of all we must realise that $a^x = e^{x\log a}$, so that $a^x - 1 \sim e^{x\log a} - 1 \sim x\log a$ when $x \to 0$. Then

$$\ell_a = \lim_{x\to 0} \frac{e^{x\log a} - 1}{x} = \lim_{x\to 0} \frac{\cancel{x}\log a}{\cancel{x}} = \log a.$$

Therefore $\ell = \log a - \log b = \log(a/b)$.

Problem 5.3

(i) On the one hand, as $x \to \infty$,

$$x^3 + 4x - 7 = x^3\left(1 + \frac{4}{x^2} - \frac{7}{x^3}\right) \sim x^3.$$

On the other hand,

$$7x^2 - \sqrt{2x^6 + x^5} = 7x^2 - x^3\sqrt{2 + \frac{1}{x}} = x^3\left(\frac{7}{x} - \sqrt{2 + \frac{1}{x}}\right) \sim -\sqrt{2}x^3.$$

Therefore

$$\lim_{x\to\infty} \frac{x^3 + 4x - 7}{7x^2 - \sqrt{2x^6 + x^5}} = \lim_{x\to\infty} \frac{\cancel{x^3}}{-\sqrt{2}\cancel{x^3}} = -\frac{1}{\sqrt{2}}.$$

(ii) On the one hand, as $x \to \infty$,

$$x + \sin x^3 = x\left(1 + \frac{\sin x^3}{x}\right) \sim x$$

because $|\sin x^3| \leqslant 1$ for all $x \in \mathbb{R}$. On the other hand,

$$5x + 6 \sim 5x.$$

Therefore

$$\lim_{x\to\infty} \frac{x + \sin x^3}{5x + 6} = \lim_{x\to\infty} \frac{\cancel{x}}{5\cancel{x}} = \frac{1}{5}.$$

(iii) As $x \to \infty$,

$$\sqrt{x + \sqrt{x + \sqrt{x}}} = \sqrt{x}\sqrt{1 + \frac{1}{x}\sqrt{x + \sqrt{x}}} = \sqrt{x}\sqrt{1 + \sqrt{\frac{1}{x} + \frac{1}{x^{3/2}}}} \sim \sqrt{x},$$

thus

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}} = \lim_{x\to\infty} \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = 1.$$

(iv) This is an indeterminacy $\infty - \infty$, so we must transform

$$\sqrt{x^2 + 4x} - x = \frac{\left(\sqrt{x^2 + 4x} - x\right)\left(\sqrt{x^2 + 4x} + x\right)}{\sqrt{x^2 + 4x} + x} = \frac{x^2 + 4x - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Now, as $x \to \infty$,

$$\sqrt{x^2 + 4x} + x = x\left(\sqrt{1 + \frac{4}{x}} + 1\right) \sim 2x,$$

therefore

$$\lim_{x\to\infty}\left(\sqrt{x^2 + 4x} - x\right) = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2 + 4x} + x} = \lim_{x\to\infty} \frac{4\cancel{x}}{2\cancel{x}} = 2.$$

(v) To begin with, $e^x - 1 \sim e^x$ as $x \to \infty$, so

$$\lim_{x\to\infty} \frac{e^x}{e^x-1} = \lim_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}} = 1.$$

Now, since $\lim\limits_{x\to-\infty} e^x = 0$,

$$\lim_{x\to-\infty} \frac{e^x}{e^x-1} = 0.$$

(vi) When the sign of x is not defined —as in this case that we need to calculate the two limits when $x \to \pm\infty$— we can write

$$\sqrt{4x^2+1} = 2|x|\sqrt{1+\frac{1}{4x^2}}.$$

So as $x \to \pm\infty$

$$\sqrt{4x^2+1} \sim 2|x|.$$

Then,

$$\lim_{x\to\infty} \frac{x-2}{4x^2+1} = \lim_{x\to\pm\infty} \frac{\cancel{x}}{2\cancel{x}} = \frac{1}{2}, \qquad \lim_{x\to-\infty} \frac{x-2}{4x^2+1} = \lim_{x\to\pm\infty} \frac{\cancel{x}}{-2\cancel{x}} = -\frac{1}{2}.$$

(vii) We can express $\tanh x$ in different ways. Each one will be more suitable to calculate one specific limit. Thus, multiplying or dividing numerator and denominator by e^x,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} = \frac{1-e^{-2x}}{1+e^{-2x}}.$$

So

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^{2x}-1}{e^{2x}+1} = \lim_{x\to\infty} \frac{\cancel{e^{2x}}}{\cancel{e^{2x}}} = 1,$$

and

$$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \lim_{x\to-\infty} \frac{-\cancel{e^{-2x}}}{\cancel{e^{-2x}}} = -1,$$

(viii) We can rewrite

$$\frac{e^x}{\sinh x} = \frac{2e^x}{e^x - e^{-x}} = \frac{2e^{2x}}{e^{2x}-1} = \frac{2}{1-e^{-2x}},$$

so,

$$\lim_{x\to\infty} \frac{e^x}{\sinh x} = \lim_{x\to\infty} \frac{2e^{2x}}{e^{2x}-1} = \lim_{x\to\infty} \frac{2\cancel{e^{2x}}}{\cancel{e^{2x}}} = 2,$$

and

$$\lim_{x\to-\infty} \frac{e^x}{\sinh x} = \lim_{x\to-\infty} \frac{2}{1-e^{-2x}} = \lim_{x\to-\infty} \frac{2}{-e^{-2x}} = 0.$$

(ix) We are facing here an indeterminacy 1^∞, therefore

$$\lim_{x\to\pm\infty}\left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}=e^c,$$

where

$$c=\lim_{x\to\pm\infty}\sqrt{4x^2+x-3}\left(\frac{2x+7}{2x-6}-1\right)=\lim_{x\to\pm\infty}\frac{13\sqrt{4x^2+x-3}}{2x-6}.$$

But $\sqrt{4x^2+x-3}\sim 2|x|$ as $x\to\pm\infty$, we have

$$\lim_{x\to\infty}\frac{13\sqrt{4x^2+x-3}}{2x-6}=\lim_{x\to\infty}\frac{26\cancel{x}}{2\cancel{x}}=13,\qquad \lim_{x\to-\infty}\frac{13\sqrt{4x^2+x-3}}{2x-6}=\lim_{x\to-\infty}\frac{-26\cancel{x}}{2\cancel{x}}=-13.$$

Problem 5.4

(i) If x is a *positive* number close to 0 we have $\lfloor x\rfloor=0$. If it is *negative,* $\lfloor x\rfloor=-1$. Therefore

$$\lim_{x\to0^+}\left(\frac{1}{x}\right)^{\lfloor x\rfloor}=\lim_{x\to0^+}\left(\frac{1}{x}\right)^{0}=\lim_{x\to0^+}1=1,$$

and

$$\lim_{x\to0^-}\left(\frac{1}{x}\right)^{\lfloor x\rfloor}=\lim_{x\to0^-}\left(\frac{1}{x}\right)^{-1}=\lim_{x\to0^-}x=0.$$

(ii) We will change the variable x to $t=1/x$:

$$\lim_{x\to0^+}e^{1/x}=\lim_{t\to+\infty}e^t=\infty,\qquad \lim_{x\to0^-}e^{1/x}=\lim_{t\to-\infty}e^t=0.$$

(iii) We will change the variable x to $t=1/x$:

$$\lim_{x\to0^+}\frac{1-e^{1/x}}{1+e^{1/x}}=\lim_{t\to+\infty}\frac{1-e^t}{1+e^t}=\lim_{t\to+\infty}\frac{-\cancel{e^t}}{\cancel{e^t}}=-1,$$

and

$$\lim_{x\to0^-}\frac{1-e^{1/x}}{1+e^{1/x}}=\lim_{t\to-\infty}\frac{1-e^t}{1+e^t}=1.$$