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Calculus I

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Unit 6. Continuity

Exercises



Problems

Problem 6.1

- (a) Prove that if f is continuous then so is $|f|$. Show that the reciprocal is false by finding a counterexample.
- (b) What can be said about a function that is continuous but all the values it takes are in \mathbb{Q} ?

Problem 6.2

- (a) Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous, surjective function. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$.
- (b) Let f be a continuous function in $[a, b]$ and let $x_1, \dots, x_n \in [a, b]$. Prove that there exists $c \in [a, b]$ such that $f(c) = \frac{1}{n} \sum_{k=1}^n f(x_k)$.

Problem 6.3

Consider the function

$$f(x) = \frac{1}{\lambda x^2 - 2\lambda x + 1}.$$

Determine for which values $\lambda \in \mathbb{R}$ the function is continuous in (a) \mathbb{R} , or (b) $[0, 1]$.

Problem 6.4

Study the continuity of the following functions:

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| (i) $f(x) = \frac{e^{-5x} + \cos x}{x^2 - 8x + 12};$ | (x) $f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ -x, & x \notin \mathbb{Q}; \end{cases}$ |
| (ii) $f(x) = e^{3/x} + x^3 - 9;$ | (xi) $f(x) = \begin{cases} \sin(\pi x), & x < -1, \\ x - x, & -1 \leq x < 1, \\ (x-1)^3, & x \geq 1; \end{cases}$ |
| (iii) $f(x) = x^3 \tan(3x + 2);$ | (xii) $f(x) = \begin{cases} (x+1)^2, & x \leq -1, \\ \operatorname{sgn} x + 1, & -1 < x < 1, \\ 2x, & x \geq 1; \end{cases}$ |
| (iv) $f(x) = \sqrt{x^2 - 5x + 6};$ | (xiii) $f(x) = \begin{cases} x^2, & x \leq -2, \\ x^2 - 1 , & -2 < x < 2, \\ 4x - 5, & x \geq 2; \end{cases}$ |
| (v) $f(x) = (\arcsin x)^3;$ | (xiv) $f(x) = \begin{cases} (x-1)^2, & x > 1, \\ x - [x], & -1 \leq x \leq 1, \\ x + 1, & x < -1. \end{cases}$ |
| (vi) $f(x) = (x-5) \log(8x-3);$ | |
| (vii) $f(x) = x - [x];$ | |
| (viii) $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0; \end{cases}$ | |
| (ix) $f(x) = \begin{cases} \frac{\tan x}{\sqrt{x}}, & x > 0, \\ 0, & x = 0, \\ e^{1/x}, & x < 0; \end{cases}$ | |

Problem 6.5

Which of these equations have at least one solution in the specified set?:

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| (i) $x^2 - 18x + 2 = 0$, in $[-1, 1]$; | (v) $f(x) = 0$, in $[-2, 2]$, where f is given by |
| (ii) $x - \sin x = 1$, in \mathbb{R} ; | $f(x) = \begin{cases} x^2 + 2, & -2 \leq x < 0, \\ -(x^2 + 2), & 0 \leq x \leq 2; \end{cases}$ |
| (iii) $e^x + 1 = 0$, in \mathbb{R} ; | (vi) $\frac{1}{4}x^3 - \sin(\pi x) + 3 = \frac{7}{3}$, in $[-2, 2]$; |
| (iv) $\cos x + 2 = 0$, in \mathbb{R} ; | (vii) $ \sin x = \sin x + 3$, in \mathbb{R} . |

Problem 6.6

Prove that any polynomial of odd degree has at least one real root.